Math 510 - Fall 2006
Algebraic Curves and Riemann Surfaces

Problem Set 2 - Due Tuesday October 3

Notation:

- Σ and Σ' are closed Riemann surfaces,
- $\mathcal{HOL}(\Sigma)$ (resp. $K(\Sigma)$) denotes the holomorphic (resp. meromorphic) functions on Σ
- $\mathcal{HOL}(\Sigma, \Sigma')$ denotes the holomorphic maps from Σ to Σ'.

Question 1

In class we showed the correspondence between meromorphic functions $f(z)$ with a single pole at $z = 0$, and holomorphic maps $F : \mathbb{C} \longrightarrow \mathbb{P}^1$ with $F^{-1}([0,1]) = \{0\}$.

1. Prove the general version of this result, i.e. prove that a meromorphic function with poles at $\{p_1, p_2, \ldots, p_l\}$ is equivalent to a holomorphic map (from $\mathbb{C}$ to $\mathbb{P}^1$) with $F^{-1}([0,1]) = \{p_1, p_2, \ldots, p_l\}$.
2. Prove that for any Riemann surface Σ,

$$\mathcal{HOL}(\Sigma, \mathbb{P}^1) = K(\Sigma) \ .$$

Question 2

(1) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic map such that $f(0) = 0$. Show that there is always a biholomorphism, say ϕ, defined on a neighborhood of 0 such that

$$f(\phi(z)) = z^\nu$$

for some positive integer $\nu \in \mathbb{Z}$.

(2) Let $F : \Sigma \longrightarrow \Sigma'$ be a holomorphic map. Suppose that $F(p) = q$. With respect to local coordinates in which $z(p) = 0$ and $z'(q) = 0$, F is then given by the map

$$z \mapsto z^\mu h(z) \ ,$$

where ν is a positive integer and $h(z)$ is holomorphic and non-vanishing at $z = 0$. Prove that the integer μ is independent of the choice of local coordinates. Also show that the coordinates can be chosen such that the map F is given by the map

$$z \mapsto z^\mu \ .$$

Define the **ramification index of F at p** to be

$$R_p(f) = \mu - 1 \ .$$

(3) Let $f \in K(\Sigma)$ be a meromorphic function. For each point $p \in \Sigma$ we can pick local coordinates (denoted by z) in a neighborhood of p such that $z(p) = 0$. With respect to these coordinates f is then given by the map

$$z \mapsto z^\nu h(z) \ ,$$

where $h(z)$ is holomorphic and non-vanishing at $z = 0$. Prove that the integer ν is independent of the choice of local coordinates. Also show that the coordinates can be chosen such that f is given by the map

$$z \mapsto z^\nu \ .$$

Define the **order (or multiplicity) of f at p** to be

$$\nu_p(f) = \nu \ .$$

(4) Under the correspondence between $\mathcal{HOL}(\Sigma, \mathbb{P}^1)$ and $K(\Sigma)$, suppose that $F \in \mathcal{HOL}(\Sigma, \mathbb{P}^1)$ is the holomorphic map corresponding to $f \in K(\Sigma)$, i.e.

$$F(p) = \begin{cases} [0,1] \text{ if } f \text{ has a pole at } p \\ [1, f(p)] \text{ otherwise} \end{cases}$$

Prove that

$$R_p(F) = \begin{cases} \nu_p(f) - 1 \quad \text{if f(p)=0} \\ -(\nu_p(f) + 1) \quad \text{if } f \text{ has a pole at } p \\ \nu_p(f - f(p)) - 1 \quad \text{otherwise} \end{cases}$$

Question 3

(1) Let $ZP(f)$ denote the set of zeros and poles of a meromorphic function $f \in K(\Sigma)$. Prove that $ZP(f)$ is a discrete set and hence, if Σ is compact, is a finite collection of points.

(2) Let F be a holomorphic map between Riemann surfaces Σ and Σ'. Prove that the set of points $p \in \Sigma$ at which the ramification index $R_p(F) \geq 1$ forms a discrete subset of Σ.

Question 4

Prove that every non-constant meromorphic function on a closed Riemann surface has at least one zero and one pole.

Question 5

(a) Let $f(z, w)$ and $g(z, w)$ be homogeneous polynomials of the same degree in (z, w). Prove that the map

$$[z, w] \mapsto [f(z, w), g(z, w)]$$

is a well defined holomorphic map from $\mathbb{CP}^1$ to $\mathbb{CP}^1$ if and only if f and g have no common factor.

(b) Suppose that $f(z, w)$ and $g(z, w)$ are both degree 1, say

$$\begin{aligned} f(z, w) &= az + bw \\ g(z, w) &= cz + dw \ , \end{aligned}$$

and let $F_A : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ be the map defined by

$$[z, w] \mapsto [az + bw, cz + dw]$$

Show that F_A is a well defined biholomorphic map if and only if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an invertible 2×2 matrix over $\mathbb{C}$. For which A is F_A the identity?

(c) Let $F_A : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ be as in (b), and let $f \in K(\mathbb{CP}^1)$ be the corresponding meromorphic function on $\mathbb{CP}^1$. Find the rational function corresponding to f (under the correspondence between rational functions on $\mathbb{C}$ and meromorphic functions on $\mathbb{CP}^1$).

Question 6

Define the **degree** of a holomorphic map $F : \Sigma \longrightarrow \Sigma'$ to be the integer

$$deg(F) = \sum_{F(p)=q} (R_p(F) + 1) \ ,$$

where $q \in \Sigma'$ is any point. Let f be the meromorphic function on $\mathbb{CP}^1$ defined by the rational function

$$z \mapsto \frac{z^3}{(1 - z^2)} \ ,$$

and let $F : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ be the corresponding holomorphic map. Compute the degree of F.