# MATH 231 U1, Spring 2009
## Answers to HW 10, Section 6.6 Worksheet
## Due Friday February 13th, 2009

The assignment was just to turn in the answers to question 4 (a)-(e). Here I will give answers (or at least hints) to all the questions.

#4. Determine if the following integrals are convergent or divergent. (You do not have to calculate their values.)

(a) $\int_1^\infty \frac{1}{x^{29}} dx$ (b) $\int_0^3 \frac{1}{x^{29}} dx$ (c) $\int_{492}^\infty \frac{1}{\sqrt{x^{2.5}}} dx$ (d) $\int_1^\infty \frac{x^2-1}{x^4+3x+2} dx$

(e) $\int_2^\infty \frac{3}{\ln x}\, dx$

ANSWERS

(a) $\int_1^\infty \frac{1}{x^{29}} dx$ converges by the "p-test" from problem 1, since $29 > 1$

(b) $\int_0^3 \frac{1}{x^{29}} dx$ diverges (by problem 35 from section 6.6), since $29 > 1$

(c) $\int_{492}^\infty \frac{1}{\sqrt{x^{2.5}}} dx = \int_{492}^\infty x^{-1.25} dx$ diverges by the p-test" from problem 1, since $1.25 > 1$

(d) $\int_1^\infty \frac{x^2-1}{x^4+3x+2} dx$

For $x \geq 1$, clearly $x^4 + 3x + 2 \geq x^4$. And, $x^2 - 1 \leq x^2$, so

$$\frac{x^2-1}{x^4+3x+2} \leq \frac{x^2}{x^4+3x+2} \leq \frac{x^2}{x^4} = \frac{1}{x^4}$$

We know that $\int_1^\infty \frac{1}{x^4}\, dx$ converges since $4 > 1$. Thus, by the comparison test, $\int_1^\infty \frac{x^2-1}{x^4+3x+2} dx$ converges as well.

(e) $\int_2^\infty \frac{3}{\ln x}\, dx$

For $x \geq 2$ we know that $\ln x \leq x$. So,

$$\frac{3}{\ln x} \geq \frac{3}{x} \geq \frac{1}{x}$$

We know that $\int_2^\infty \frac{1}{x}\, dx$ diverges, so by the comparison test $\int_2^\infty \frac{3}{\ln x}\, dx$ diverges too.

#1. The "$p$-test for integrals" states that $\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$ converges for $p > 1$ and diverges otherwise. (Note: compare this to problem 35 from your section 6.6 homework)
In this exercise we will prove this.
a) Let $p = 1$. Show that the integral diverges.
b) Let $p < 1$. Without explicitly calculating the integrals, prove that the integral diverges.
(Hint: you just showed $\int_1^\infty \frac{1}{x}\,dx$ diverges, now you can use a comparison.)
c) Let $p > 1$. Show that the integral converges to $\frac{1}{p-1}$.

ANSWER

(a) When $p = 1$,

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \int_1^\infty x^{-1}\,dx \\
&= \lim_{R\to\infty} \int_1^R x^{-1}\,dx \\
&= \lim_{R\to\infty} \Big[\ln x\Big]_1^R \\
&= \lim_{R\to\infty} \Big[\ln R - \ln 1\Big] = \infty
\end{aligned}$$

So the integral diverges in this case.

(b) Let $p < 1$. Then, for all $x \in [1, \infty)$ we know that $x^p \leq x$. (It is $\leq$ because when $x = 1$ these are $=$).

Thus, $\frac{1}{x^p} \geq \frac{1}{x}$. Now, because $\int_1^\infty x^{-1}\,dx$ diverges, we may conclude that for $p < 1$

$\int_1^\infty \frac{1}{x^p}\,dx$ diverges too, by the comparison test.

(c) Let $p > 1$. For this one we actually have to calculate the value of the integral, so we will start from the definition of an improper integral.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{R\to\infty} \int_1^R x^{-p}\,dx \\
&= \lim_{R\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^R \\
&= \lim_{R\to\infty} \left[\frac{R^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right] \\
&= \lim_{R\to\infty} \left[\frac{R^{1-p}}{1-p}\right] - \frac{1}{1-p} \\
&= 0 - \frac{1}{1-p} = \frac{1}{p-1}
\end{aligned}$$

The limit $\lim_{R\to\infty} \left[\frac{R^{1-p}}{1-p}\right] = 0$ because $p > 1$, so the $R$ in the numerator is taken to a negative power, so the limit of the numerator is 0, and the limit of the denominator is $1-p \neq 0$, so the limit of the quotient is 0.

#2. (a) We have learned the comparison test for integrals of the form $\int_a^\infty f(x)\,dx$ (Theorem 6.1 from Section 6.6). Review this theorem, then, come up with a statement for a comparison test for integrals of the form $\int_0^a f(x)\,dx$

The statement should start analogously to Theorem 6.1:
Let $f$ and $g$ be continuous on $(0,1]$ and $0 \leq f(x) \leq g(x)$ for all $x \in (0,1]$...

(b) Use your comparison test for integrals on $[0,1]$ to figure out the following question:
Does $\int_0^1 \frac{\sin^2 x}{x^{1/4}}\,dx$ converge or diverge? Justify your answer!

ANSWER

(a) The comparison test should read something like: Let $f$ and $g$ be continuous on $(0,1]$ and $0 \leq f(x) \leq g(x)$ for all $x \in (0,1]$ and let $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^+} g(x) = \infty$.

(i) If $\int_0^1 g(x)\,dx$ converges, then $\int_0^1 f(x)\,dx$ converges.

(ii) If $\int_0^1 f(x)\,dx$ diverges, then $\int_0^1 g(x)\,dx$ diverges.

(b) Here, we will compare $\int_0^1 \frac{\sin^2 x}{x^{1/4}}\,dx$ to $\int_0^1 \frac{1}{x^{1/4}}\,dx$. Since $0 \leq \sin^2 x \leq 1$, we know that

$$\frac{\sin^2 x}{x^{1/4}} \leq \frac{1}{x^{1/4}}.$$

Therefore, if we can show that $\int_0^1 \frac{1}{x^{1/4}}\,dx$ converges, we will know that $\int_0^1 \frac{\sin^2 x}{x^{1/4}}\,dx$ converges too.

$\int_0^1 \frac{1}{x^{1/4}}\,dx$ converges by problem #35 from Section 6.6. (That problem says that integrals of the form $\int_0^1 \frac{1}{x^s}\,dx$ converge if and only if $s < 1$.) If you have forgotten this fact, you could also easily calculate the improper integral to show it converges.

So, by our new version of the comparison test, we know that $\int_0^1 \frac{\sin^2 x}{x^{1/4}}\,dx$ converges.

#3. Say you know $f$ and $g$ are both continuous on $[a, \infty)$, and that there exists a number $c$ in $[a, \infty)$ such that for all $x \geq c$, $0 \leq f(x) \leq g(x)$.

(a) Say you know $\displaystyle\int_a^\infty g(x)\ dx$ converges. What can you say about $\displaystyle\int_a^\infty f(x)\ dx$?

(b) Say you know $\displaystyle\int_a^\infty f(x)\ dx$ diverges. What can you say about $\displaystyle\int_a^\infty g(x)\ dx$?

(c) Use your conclusions above to determine whether $\displaystyle\int_1^\infty \frac{1}{4x^2 - 12}\ dx$ converges or diverges. (Hint: compare to $\frac{1}{x^2}$)

ANSWER

Because we may write

$$\int_a^\infty f(x)\ dx = \int_a^c f(x)\ dx + \int_c^\infty f(x)\ dx$$

and $f(x)$ is continuous on $[a, c]$, we know $\int_a^c f(x)\ dx$ is a normal, (not improper) integral, and that the whether or not $\int_a^\infty f(x)\ dx$ converges only depends on whether or not $\int_c^\infty f(x)\ dx$ converges.

So, in this situation, when we are figuring out whether an improper integral on $[a, \infty)$ converges or diverges we can basically disregard everything between $a$ and $c$. This explains the answers below.

(a) You can conclude that $\displaystyle\int_a^\infty f(x)\ dx$ converges

(b) You can conclude that $\displaystyle\int_a^\infty g(x)\ dx$ diverges

(c) This question turned out to involve more than just what we investigated in parts (a) and (b), because in addition to having a horizontal asymptote as $x \to \infty$, this function $\frac{1}{4x^2-12}$ has a vertical asymptote at $\sqrt{3}$. So, one way to approach this problem split up the integral into 3 pieces:

$$\int_1^{\sqrt{3}} \frac{1}{4x^2 - 12}\ dx + \int_{\sqrt{3}}^2 \frac{1}{4x^2 - 12}\ dx + \int_2^\infty \frac{1}{4x^2 - 12}\ dx$$

The second integral is where we can apply part (a) above, because we can show that for $x \geq 2$,

$$x^2 \leq 4x^2 - 12$$

which lets us conclude that for $x \geq 2$

$$\frac{1}{x^2} \geq \frac{1}{4x^2 - 12}.$$

We know that $\displaystyle\int_{\sqrt{3}}^\infty \frac{1}{x^2}\ dx$ converges, and part (a) now tells us that $\displaystyle\int_{\sqrt{3}}^\infty \frac{1}{4x^2 - 12}\ dx$ converges too.

I'll let you handle the other two improper integrals.