MATH 231 U1, Spring 2009
Answers to HW 12, Section 8.1, 10, 12, 16, 23, 26
Due Monday February 23rd, 2009

#10. (a) Does $a_n = \dfrac{4}{\sqrt{n+1}}$ converge? If so, to what?
(b) Use the definition to show that the sequence converges. (c) Plot or draw a picture.

ANSWER

(a) Yes, $\lim_{n\to\infty} \dfrac{4}{\sqrt{n+1}} = 0$. We can see this because $\sqrt{n+1}$ goes to ∞, while 4 is just a constant.

(b) In order to prove that $\lim_{n\to\infty} \dfrac{4}{\sqrt{n+1}} = 0$, we must prove that for any $\epsilon > 0$, there is an integer N so that for all $n > N$,

$$|\frac{4}{\sqrt{n+1}} - 0| < \epsilon.$$

That is, find N so that for $n > N$,

$$|\frac{4}{\sqrt{n+1}}| = \frac{4}{\sqrt{n+1}} < \epsilon.$$

Below, we try use algebra to get an idea of how big n must be in order for $\frac{4}{\sqrt{n+1}} < \epsilon$ to be true:

$$\frac{4}{\sqrt{n+1}} < \epsilon$$

$$\frac{16}{n+1} < \epsilon^2$$

$$16 < \epsilon^2(n+1)$$

$$16 < \epsilon^2 n + \epsilon^2$$

$$16 - \epsilon^2 < \epsilon^2 n$$

$$\frac{16-\epsilon^2}{\epsilon^2} < n$$

So, we choose $N \geq \frac{16-\epsilon^2}{\epsilon^2}$, and then we know that for $n > N \geq \frac{16-\epsilon^2}{\epsilon^2}$

$$n > \frac{16 - \epsilon^2}{\epsilon^2}$$

$$n\epsilon^2 > 16 - \epsilon^2$$

$$n\epsilon^2 + \epsilon^2 > 16$$

$$\epsilon^2(n + 1) > 16$$

$$\epsilon^2 > \frac{16}{n + 1}$$

$$\epsilon > \frac{4}{\sqrt{n + 1}}$$

This last step where we take the square root works because we know $\epsilon > 0$ by our assumption, so $\sqrt{\epsilon^2} = \epsilon$.

(c) You guys can draw a picture or use a calculator or computer.

#12. Does $a_n = \dfrac{5n^3 - 1}{2n^3 + 1}$ converge or diverge?

ANSWER

Divide the top and the bottom by n^3 to rearrange and see what the limit will be:

$$\lim_{n\to\infty} \frac{5n^3 - 1}{2n^3 + 1} = \lim_{n\to\infty} \frac{5 - \frac{1}{n^3}}{2 + \frac{1}{n^3}} = \frac{5}{2}$$

#16. Does $a_n = (-1)^n \dfrac{n + 4}{n + 1}$ converge or diverge?

ANSWER

It diverges. The even terms have a limit of 1 and the odd terms have a limit of -1, so overall, the sequence diverges.

#23. Does $a_n = \dfrac{n2^n}{3^n}$ converge or diverge?

ANSWER

We will look at the limit of the associated function $f(x) = \dfrac{x2^x}{3^x}$ (notice that our sequence $a_n = f(n)$ for integers $n \geq 0$.)

We may find the limit $\lim_{x\to\infty} \frac{x2^x}{3^x}$ by doing some rearranging and then applying L'Hopital's rule.

$$\lim_{x\to\infty} \frac{x2^x}{3^x} = \lim_{x\to\infty} x(\frac{2}{3})^x = \lim_{x\to\infty} \frac{x}{(\frac{3}{2})^x}$$

We may now apply L'Hopital's Rule, because the limit of the denominator and the limit of the numerator for $\lim_{x\to\infty} \frac{x}{(\frac{3}{2})^x}$ are both ∞:

$$\lim_{x\to\infty} \frac{x}{(\frac{3}{2})^x} = \lim_{x\to\infty} \frac{1}{\ln(3/2)(\frac{3}{2})^x} = 0$$

So,

$$\lim_{x\to\infty} \frac{x2^x}{3^x} = 0$$

Now, since the limit of our function $\lim_{x\to\infty} \frac{x2^x}{3^x} = 0$, we know by Theorem 1.2 in Section 8.1 that $\lim_{n\to\infty} \frac{n2^n}{3^n}$ must also converge to 0.

#26. Does $a_n = \sqrt{n^2+n} - n$ converge or diverge?

ANSWER

We employ an algebra trick, where we multiply (and divide) by the conjugate in order to simplify the numerator. The last step is dividing the top and bottom by n:

$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} (\sqrt{n^2+n} - n)\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n}$$

$$= \lim_{n\to\infty} \frac{(n^2+n-n^2)}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$$

So, this sequence converges to $\frac{1}{2}$.