

MATH 231 U1, Spring 2009  
Answers to HW 13, Section 8.1, problems 24, 32, 40, 57  
Due Wednesday February 25th, 2009

#24. Does  $a_n = \frac{\cos n}{e^n}$  converge or diverge?

ANSWER

$$-1 \leq \cos n \leq 1 \text{ implies that } \frac{-1}{e^n} \leq \frac{\cos n}{e^n} \leq \frac{1}{e^n}.$$

It is easy to see that  $\lim_{n \rightarrow \infty} \frac{1}{e^n} = 0$  because  $e^n$  is going to  $\infty$  and 1 is a constant. Likewise,  
 $\lim_{n \rightarrow \infty} \frac{-1}{e^n} = 0$ .

Therefore, by the Squeeze Theorem,  $\frac{\cos n}{e^n}$  converges to 0 too.

#32. Use the Squeeze Theorem to prove  $a_n = (-1)^n \frac{\ln n}{n^2}$  converges to 0. (You may use the facts that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  and  $\lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$ .)

ANSWER

For  $n \geq 1$ ,

$$\frac{\ln n}{n^2} \leq \frac{n}{n^2} = \frac{1}{n}.$$

This inequality is true because for  $n \geq 1$ ,  $\ln n \leq n$ .

From this inequality we get that for  $n \geq 1$

$$\frac{-1}{n} \leq (-1)^n \frac{\ln n}{n^2} \leq \frac{1}{n}.$$

We know that the limits of the sequences  $a_n = \frac{1}{n}$  and  $b_n = \frac{-1}{n}$  are both 0, therefore by the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} (-1)^n \frac{\ln n}{n^2} = 0.$$

#40. Show the sequence  $a_n = \frac{6n-1}{n+3}$  is bounded.

ANSWER

For  $n \geq 1$

$$\left| \frac{6n-1}{n+3} \right| = \frac{6n-1}{n+3} \quad (1)$$

$$< \frac{6n}{n+3} \quad (2)$$

$$< \frac{6n}{n} = 6 \quad (3)$$

From line (1) to line (2), the inequality holds because  $6n-1 < 6n$ , and from line (2) to line (3) the inequality holds because  $n+3 > n$ . So, 6 is a bound for this sequence.

# 57. Define  $a_n = \frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2}$ . Evaluate the sum using a formula from Section 4.2 and show the sequence converges. Thinking of  $a_n$  as a Riemann sum, identify the definite integral to which the sequence converges.

ANSWER

Notice that

$$a_n = \frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2} = \frac{1}{n^2}(1 + 2 + 3 + \dots + n).$$

In Section 4.2 we learned that the sum of the 1st  $n$  integers is given by

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Therefore,

$$a_n = \frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n}{n^2} = \frac{1}{n^2}(1 + 2 + 3 + \dots + n) = \frac{1}{n^2} \left( \frac{n(n+1)}{2} \right)$$

So,

$$a_n = \frac{1}{n^2} \left( \frac{n(n+1)}{2} \right) = \frac{n^2 + n}{2n^2} = \frac{1}{2} + \frac{1}{2n}$$

and therefore

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{2} + \frac{1}{2n} = \frac{1}{2}.$$

So the sequence converges to  $\frac{1}{2}$ .

Now, for each integer  $n$ , we want to think of  $a_n$  as a Riemann sum with  $n$ -rectangles:

$$a_n = \sum_{i=1}^n \frac{i}{n^2} = \sum_{i=1}^n \frac{i}{n} \frac{1}{n}$$

Working backwards, we can see that this is a Riemann sum estimating the area between  $f(x) = x$  and the  $x$ -axis, on the interval  $[0, 1]$  with  $\Delta x = \frac{1}{n}$  and right endpoints  $\frac{i}{n}$ . (Note, this is not the only choice of function, interval and evaluation points which will work, but it is a logical one.)

So, taking limits we get

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i}{n} \frac{1}{n} = \int_0^1 x \, dx$$

by the definition of the definite integral as a limit of a Riemann sum.

This matches our earlier answer, because you can easily calculate that  $\int_0^1 x \, dx = \frac{1}{2}$

As a bonus example of how to show if a sequence is monotone, here is how to do

# 36. Determine whether  $a_n = \frac{n!}{5^n}$  is increasing, decreasing or neither.

ANSWER Since this sequence is always positive, we can look at  $\frac{a_{n+1}}{a_n}$  to see whether this is  $\geq 1$  for large enough  $n$ 's or not. If it is  $\geq 1$  for large enough  $n$ 's, then the sequence is increasing after some point, because  $\frac{a_{n+1}}{a_n} \geq 1$  means that  $a_{n+1} \geq a_n$ .

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{5^{n+1}}}{\frac{n!}{5^n}} = \frac{(n+1)! 5^n}{5^{n+1} n!} = \frac{n+1}{5}$$

Now, for  $n \geq 5$ , we know that  $\frac{a_{n+1}}{a_n} = \frac{n+1}{5} > 1$ , so for  $n \geq 5$ , the sequence is increasing.

So, overall the sequence is not increasing, but once we pass  $n = 5$  it is always increasing. This is relevant because, in order to find the limit of a sequence, it doesn't really matter what the first few terms do. So, we may ignore finitely many terms at the beginning of the sequence.