MATH 231 U1, Spring 2009

Answers to HW 14, Section 8.2, Problems 2, 6, 8, 12, 13, 18, 43

Due Monday March 2, 2009

For all these problems, determine whether the series converges or diverges. If it converges, find the sum of the series.

#2.

$$\sum_{k=0}^{\infty} \frac{1}{3}(5)^k$$

ANSWER

This is a geometric series, with $a = \frac{1}{3}$ and ratio $r = 5$. Since the $|r|$ is greater than 1, the series diverges.

#6.

$$\sum_{k=0}^{\infty} 5\Big(-\frac{1}{3}\Big)^k$$

This is a geometric series, with $a = 5$ and ratio $r = -\frac{1}{3}$. Since $|r| = \frac{1}{3}$ is less than 1, the series converges to $\dfrac{a}{1-r} = \dfrac{5}{1+\frac{1}{3}} = 5\dfrac{3}{4} = \dfrac{15}{4}$.

#8.

$$\sum_{k=1}^{\infty} \frac{k}{k+2}$$

ANSWER

It is always a good idea to start with the k^{th}-term test for divergence. To do this, calculate

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+2} = \lim_{k\to\infty} \frac{1}{1+\frac{2}{k}} = 1$$

Since this limit $\neq 0$, we know the series must diverge by the k^{th}-term test.

#12.

$$\sum_{k=0}^{\infty} \frac{4}{k+1}$$

ANSWER

If we shift the indices of this sum by one we get

$$\sum_{k=0}^{\infty} \frac{4}{k+1} = \sum_{k=1}^{\infty} \frac{4}{k}$$

(If you are skeptical, write out some terms of both series to see this is true!)
Then, we see

$$\sum_{k=1}^{\infty} \frac{4}{k} = 4 \sum_{k=1}^{\infty} \frac{1}{k}$$

which is 4 times the harmonic series, so it diverges.

#13.

$$\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2}$$

ANSWER

(Refer to example 2.3 for another problem like this.) First, we use partial fractions to rewrite $\frac{2k+1}{k^2(k+1)^2} = \frac{A}{k} + \frac{B}{k^2} + \frac{C}{k+1} + \frac{D}{(k+1)^2}$. Which gives

$$2k + 1 = Ak(k+1)^2 + B(k+1)^2 + Ck^2(k+1) + Dk^2$$

And solve for A, B, C, D to get $B = 1$, $D = -1$, $A = C = 0$.

So,

$$\frac{2k+1}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2}.$$

Now, if we use this to write out the nth **partial sum** of the sequence

$$S_n = (1 - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{9}) + (\frac{1}{9} - \frac{1}{16}) + ... + (\frac{1}{(n-1)^2} - \frac{1}{n^2}) + (\frac{1}{n^2} - \frac{1}{(n+1)^2}) = 1 - \frac{1}{(n+1)^2}$$

Notice that almost all of these terms cancel each other, so our nth partial sum is

$$S_n = \sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

By definition, the sum of the series is the limit of the sequence $\{S_n\}_{n=1}^{\infty}$of partial sums, so,

$$\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{(n+1)^2} = 1.$$

So, our series converges to 1. This kind of series is called a **telescoping series** because of the way the partial sums "collapse".

#18.

$$\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{1}{3^k} \right)$$

ANSWER

We break this down into its constituent pieces.

The sum $\sum_{k=0}^{\infty} \frac{1}{2^k}$ is a geometric series with $a = 1$ and $r = \frac{1}{2} < 1$. So, it converges to $\frac{a}{1-r} = \frac{1}{1-0.5} = 2$.

The sum $\sum_{k=0}^{\infty} \frac{1}{3^k}$ is a geometric series with $a = 1$ and $r = \frac{1}{3} < 1$. So, it converges to $\frac{a}{1-r} = \frac{1}{2/3} = \frac{3}{2}$.

Using Theorem 2.3, we get that

$$\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{1}{3^k} \right) = \sum_{k=0}^{\infty} \frac{1}{2^k} - \sum_{k=0}^{\infty} \frac{1}{3^k} = 2 - \frac{3}{2}$$

So the series converges to $2 - \frac{3}{2} = \frac{1}{2}$.

#43. A dosage d of a drug is given at times $t = 1, 2, 3, ...$ The drug decays exponentially with rate r in the bloodstream. The amount after $n + 1$ doses is

$$d + de^{-r} + de^{-2r} + de^{-3r} + ... + de^{-nr}.$$

Show the eventual level of the drug (after an "infinite" number of doses) is $\frac{d}{1 - e^{-r}}$.
If $r = 0.1$, find the dosage required to maintain a drug level of 2.

ANSWER

The series

$$d + de^{-r} + de^{-2r} + de^{-3r} + ... + de^{-nr} + ...$$

is a geometric series with first term $=d$ and ratio $= \frac{e^{-(k+1)r}}{e^{-kr}} = e^{-r}$. Therefore, it converges to $\frac{d}{1 - e^{-r}}$.

If $r = 0.1$, then we set $\frac{d}{1 - e^{-0.1}} = 2$ and solve for the d which maintains this drug level.

$$\frac{d}{1 - e^{-0.1}} = 2 \implies d = 2(1 - e^{-0.1})$$