MATH 231 U1, Spring 2009
Answers to HW 15, Section 8.3, Problems 10, 14, 30, 36
Due Wednesday March 4, 2009

Keep in mind for all these problems, there may be more than one way to do them!

#10.

$$\sum_{k=2}^{\infty} \frac{3}{k(\ln k)^2}$$

ANSWER

We use the Integral Test. Note that it will work even though our sum starts at 2 instead of 1, we just look at the integral from 2 to ∞ instead.

Let $f(x) = \frac{3}{x(\ln x)^2}$. Note that this function is continuous on $[2, \infty)$, and $f(x) \geq 0$ for all x in $[2, \infty)$, and it is a decreasing function on $[2, \infty)$, because the denominator $x(\ln x)^2$ is clearly increasing as $x \to \infty$.

$$\int_2^{\infty} \frac{3}{x(\ln x)^2}\, dx = \lim_{R\to\infty} \int_2^{R} \frac{3}{x(\ln x)^2}\, dx = \lim_{R\to\infty} \Big[-3(\ln x)^{-1} \Big]_2^R = -3 \lim_{R\to\infty} \frac{1}{\ln R} - \frac{1}{\ln 2} = \frac{3}{\ln 2}$$

(To find the antiderivative of $\frac{3}{x(\ln x)^2}$ you can use substitution with $u = \ln x$.) Thus, the integral converges, and by the integral test, the sum must converge too. (BUT, it DOES NOT necessarily converge to the same number!)

#14.

$$\sum_{k=4}^{\infty} \frac{\sqrt{1+\frac{1}{k}}}{k^2}$$

ANSWER

For $k \geq 4$, it is plain that

$$0 \leq 1 + \frac{1}{k} \leq 1 + \frac{1}{4} \leq 2$$

which implies

$$0 \leq \sqrt{1+\frac{1}{k}} \leq \sqrt{1+\frac{1}{4}} \leq \sqrt{2}.$$

So

$$0 \leq \frac{\sqrt{1+\frac{1}{k}}}{k^2} \leq \frac{\sqrt{1+\frac{1}{4}}}{k^2} \leq \frac{\sqrt{2}}{k^2}.$$

By the p-test, $\sum_{k=4}^{\infty} \frac{1}{k^2}$ converges. Then, by Theorem 2.3 we know that $\sqrt{2}\sum_{k=4}^{\infty} \frac{1}{k^2} = \sum_{k=4}^{\infty} \frac{\sqrt{2}}{k^2}$ also converges.

Now, by the comparison test, since $0 \leq \frac{\sqrt{1+\frac{1}{k}}}{k^2} \leq \frac{\sqrt{2}}{k^2}$, the series $\sum_{k=4}^{\infty} \frac{\sqrt{1+\frac{1}{k}}}{k^2}$ also converges.

#30.

$$\sum_{k=1}^{\infty} \frac{k+1}{k^2+2}$$

ANSWER

Here, since a direct comparison to $\frac{1}{k^2}$ is harder, we will use the Limit Comparison Test. (This is a very useful test, and one of my favorites!)

Let $a_k = \frac{k+1}{k^2+2}$ and $b_k = \frac{1}{k}$. Then

$$\frac{a_k}{b_k} = \frac{\frac{k+1}{k^2+2}}{\frac{1}{k}} = \frac{(k+1)k}{k^2+2}.$$

So, we calculate

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{(k+1)k}{k^2+2} = \lim_{k\to\infty} \frac{k^2+k}{k^2+2} = \lim_{k\to\infty} \frac{1+\frac{1}{k}}{1+\frac{2}{k^2}} = 1 > 0.$$

So, by the Limit Comparison Test, we know that $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converge or both diverge.

Since $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, we know $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k+1}{k^2+2}$ diverges too.

#36. If $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ converges, prove $\sum_{k=1}^{\infty} a_k^2$ converges.

ANSWER

Here is one way to prove it. It may not be the only way!

Assume $a_k > 0$ and $\sum_{k=1}^{\infty} a_k$ converges.

By Theorem 2.2 in Section 8.2, since $\sum_{k=1}^{\infty} a_k$ converges we know that $\lim_{k\to\infty} a_k = 0$.

By the definition of limit for a sequence, this means that there exists an N such that for all $k \geq N$, $a_k = |a_k| < 1$. (Let $\epsilon = 1$ and use the definition of limit in 8.1) (Note, this is intuitively obvious, if the limit of the sequence is 0, then eventually the terms must all be less than 1.)

Also, the fact that $\sum_{k=1}^{\infty} a_k$ converges implies that $\sum_{k=N}^{\infty} a_k$ converges too. (Think about why!)

For $k \geq N$, it is true that $0 < a_k < 1$, and therefore $0 < a_k^2 < a_k$. Then, we know by the comparison test that $\sum_{k=N}^{\infty} a_k^2$ converges too.

Lastly, this implies that $\sum_{k=1}^{\infty} a_k^2$ converges, because

$$a_1^2 + a_2^2 + ... + a_{N-1}^2 + \sum_{k=N}^{\infty} a_k^2 = \sum_{k=1}^{\infty} a_k^2$$