MATH 231 U1, Spring 2009
Homework 16 (Worksheet) Answers
Friday, March 6th, 2009

Section 8.3

1. Do Question #46 from Section 8.3:

Estimate the error in using the partial sum S_{100} to approximate the sum of the series $\sum_{k=1}^{\infty} \frac{4}{k^2}$.

ANSWER

Let $f(x) = \frac{4}{x^2}$. This function is continuous, decreasing and ≥ 0 for $x \in [1, \infty)$. So, we may use the integral test. By the Error Estimate for the Integral Test, the error $R_n = S - S_n$:

$$0 \leq R_{100} \leq \int_{100}^{\infty} \frac{4}{x^2}\, dx$$

And we calculate

$$\int_{100}^{\infty} \frac{4}{x^2}\, dx = \lim_{R\to\infty} \int_{100}^{R} \frac{4}{x^2}\, dx = \lim_{R\to\infty} \left[\frac{-4}{x}\right]_{100}^{R} = \lim_{R\to\infty} \left[\frac{-4}{x}\right]_{100}^{R} = \lim_{R\to\infty} \frac{-4}{R} + \frac{4}{100} = \frac{4}{100} = \frac{1}{25}.$$

Therefore

$$0 \leq R_{100} \leq \frac{1}{25}.$$

2. Does $\sum_{k=1}^{\infty} \frac{2^k}{1+3^k}$ converge or diverge?

ANSWER

We know that $\frac{2^k}{3^k} \geq \frac{2^k}{1+3^k} \geq 0$. We also know that $\sum_{k=1}^{\infty} \frac{2^k}{3^k}$ is a geometric series, with ratio $r = \frac{2}{3}$ and first term $a = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, it is a convergent geometric series. Then, by the comparison test we conclude that our original series $\sum_{k=1}^{\infty} \frac{2^k}{1+3^k}$ converges too.

3. Does $\sum_{k=1}^{\infty} \frac{k+1}{k2^k}$ converge or diverge? (Hint: start by splitting up $\frac{k+1}{k2^k}$ into two pieces.)

ANSWER

$$\sum_{k=1}^{\infty} \frac{k+1}{k2^k} = \sum_{k=1}^{\infty} \frac{k}{k2^k} + \frac{1}{k2^k}$$

If we consider the two parts of this separately, we see that $\sum_{k=1}^{\infty} \frac{k}{k2^k} = \sum_{k=1}^{\infty} \frac{1}{2^k}$ is a convergent geometric series (with $r = \frac{1}{2} < 1$) and $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ is convergent by the comparison test, because $0 \leq \frac{1}{k2^k} \leq \frac{1}{2^k}$. Since both of these series are convergent, the original series is convergent too. (By Theorem 2.3 from Section 8.2.)

Section 8.1

4. Show the sequence $a_n = \frac{n}{3^n}$ is monotonic and bounded (and therefore convergent).
(Hint: as in Example 1.13, start by proving the sequence is decreasing. Then, the boundedness will follow from the fact that the sequence is decreasing.)

ANSWER

We know $a_n > 0$ for all $n \geq 1$, and for $n \geq 1$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}}\frac{3^n}{n} = \frac{1}{3}\frac{n+1}{n} = \frac{1}{3}(1 + \frac{1}{n}) \leq \frac{2}{3} < 1$$

Therefore, $a_{n+1} < a_n$ for all $n \geq 1$, which means $\{a_n\}_{n=1}^{\infty}$ is a decreasing function. So, its monotonic.

Now to show $\{a_n\}_{n=1}^{\infty}$ is bounded, notice $|a_n| = a_n \leq a_1$ for all n. This is because the sequence is decreasing. So, the sequence is bounded by a_1.

Section 8.2

5. Find the sum of the series $\sum_{k=2}^{\infty} \frac{-2}{k(k-1)}$.

ANSWER

Note: the only two kinds of series we know how to find the sum for right now are geometric series and telescoping series. We could look at a ratio of terms to see that this series is not geometric. So, you might guess it is probably telescoping.

First, use a partial fraction decomposition to show

$$\frac{-2}{k(k-1)} = \frac{A}{k} + \frac{B}{k-1} = \frac{2}{k} + \frac{-2}{k-1}.$$

So, $\sum_{k=2}^{\infty} \frac{-2}{k(k-1)} = \sum_{k=2}^{\infty} \frac{2}{k} - \frac{2}{k-1}$.

Now, look at the partial sums

$$S_{n-2} = (\frac{2}{2} - \frac{2}{1}) + (\frac{2}{3} - \frac{2}{2}) + (\frac{2}{4} - \frac{2}{3}) + ... + (\frac{2}{n-1} - \frac{2}{n-2}) + (\frac{2}{n} - \frac{2}{n-1}) = -\frac{2}{1} + \frac{2}{n}$$

So, when we take the limit of this sequence of partial sums we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\frac{2}{1} + \frac{2}{n} = -2.$$