MATH 231 U1, Spring 2009
Homework 17 (8.4) Answers
Monday, March 9th, 2009

Writing Exercise #4:

A common mistake is to think that if $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ converges. Explain why this is not true for positive-term series. This is also not true for alternating series *unless* you add one more hypothesis. State the extra hypothesis and explain why it is needed.

ANSWER

The statement "if $\lim_{k\to\infty} a_k = 0$ then the series $\sum_{k=1}^{\infty} a_k$ converges" is FALSE. A counterexample which shows it is false is the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. Clearly $\lim_{k\to\infty} \frac{1}{k} = 0$, but we proved in class that this series diverges. So, it is NOT enough to show that $\lim_{k\to\infty} a_k = 0$ to show a series converges. The terms must also go to 0 "fast enough" in some sense.

The second hypothesis of the alternating series test is that for all k (or, for all large enough k)

$$0 < a_k + 1 \leq a_k.$$

To see why we need this hypothesis, consider the series described in Problem 39. Let

$$a_k = \begin{cases} \frac{1}{k} & \text{if } k \text{ is odd} \\ \frac{1}{k^2} & \text{if } k \text{ is even} \end{cases}$$

The alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is such that $\lim_{k\to\infty} a_k = 0$, but it fails to satisfy the hypothesis above, because for even values of k, $a_k \leq a_{k+1}$.

Written slightly differently, the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ from above equals $\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{(i+1)^2}\right)$. (Write out some terms to see why.) The series $\sum_{i=1}^{\infty} \frac{1}{(i+1)^2}$ converges (To show this you could, for example you could use a comparison test with $\frac{1}{i^2}$.) The series $\sum_{i=1}^{\infty} \frac{1}{i}$ diverges. So, by Theorem 2.3 from Section 8.2 the series $\sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{(i+1)^2}$ diverges. Therefore, $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ diverges, verifying that we need that second hypothesis of the AST to be true in order to conclude that an alternating series converges!

#10. Determine if the series is convergent or divergent: $\sum_{k=1}^{\infty} (-1)^k \frac{k+2}{4^k}$

ANSWER

$$\lim_{k\to\infty}\frac{k+2}{4^k}=\lim_{x\to\infty}\frac{x+2}{4^x}=\lim_{x\to\infty}\frac{1}{(\ln 4)4^x}=0$$

So, we have fulfilled that hypothesis of the AST. To check the other hypothesis, we see that clearly, $0<\frac{k+2}{4^k}$ for all k and for $k\geq 1$

$$\frac{a_{k+1}}{a_k}=\frac{k+3}{4^{k+1}}\frac{4^k}{k+2}=\frac{k+3}{4k+8}<1$$

Therefore, for $k\geq 1$ we know $a_{k+1}<a_k$. So, we've shown $0<a_{k+1}\leq a_k$.

Therefore, by the Alternating Series Test, the series $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{k+2}{4^k}$ converges.

#16. Determine if the series is convergent or divergent: $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k!}{3^k}$.

ANSWER

Look at $\displaystyle\lim_{k\to\infty}a_k=\lim_{k\to\infty}\frac{k!}{3^k}$

$$\frac{k!}{3^k}=\frac{k(k-1)(k-2)\cdot\ldots 4\cdot 3\cdot 2\cdot 1}{3\cdot 3\cdot\ldots\cdot 3}=\frac{k}{3}\frac{k-1}{3}\cdots\frac{3}{3}\frac{2}{3}\frac{1}{3}\geq\frac{6k}{27}.$$

Therefore

$$\lim_{k\to\infty}\frac{k!}{3^k}\geq\lim_{k\to\infty}\frac{6k}{27}=\infty.$$

So, the series diverges by the kth term test for divergence.

#40. Verify that the series $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ converges. It is a fact that the sum of this series is $\frac{\pi}{4}$. Given this result, we could use this series to obtain an approximation of π. How many terms would be necessary to get eight digits of π correct?

ANSWER

The series $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}$ converges by the AST because $\displaystyle\lim_{k\to\infty}\frac{1}{2k+1}=0$, and $0<\frac{1}{2k+2}\leq\frac{1}{2k+1}$ (since $2k+2>2k+1$).

Because we are given that $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}=\frac{\pi}{4}$, we know that $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{4}{2k+1}=\pi$.

We can use the error approximation for the AST to figure out how many terms of the series $\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{4}{2k+1}$ we need to approximate π to within eight digits. An error of $<10^{-7}$ will get us within eight digits.

$$|S - S_n| = |\pi - S_n| \leq a_{n+1} = \frac{4}{2n+3}$$

So, we must figure out when n is large enough that $\frac{4}{2n+3} < 10^{-7}$

$$\frac{4}{2n+1} < 10^{-7} \tag{1}$$

$$4 \times 10^7 < 2n+3 \tag{2}$$

$$\frac{40,000,000 - 3}{2} < n \tag{3}$$

$$20,000,000 - \frac{3}{2} < n \tag{4}$$

So, if we make $n = 20,000,000$, then S_n will approximate π to within eight digits.