MATH 231 U1, Spring 2009
Homework 18 (8.5) Answers
Monday, March 9th, 2009

Determine whether the series is convergent, absolutely convergent or conditionally convergent.

#10. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{2k+1}$

ANSWER

If we look at the series of absolute values, $\displaystyle\sum_{k=1}^{\infty}\frac{4}{2k+1}$, we can prove it diverges by the limit comparison test:
Let $b_k = \frac{1}{k}$. Then,

$$\lim_{k\to\infty}\frac{4}{2k+1}\frac{k}{1} = \lim_{k\to\infty}\frac{4k}{2k+1} = 2 > 0$$

So, by the limit comparison test, the series $\displaystyle\sum_{k=1}^{\infty}\frac{4}{2k+1}$ and the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ both converge or both diverge. We already know that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, therefore, $\displaystyle\sum_{k=1}^{\infty}\frac{4}{2k+1}$ diverges. So, our original series does not converge absolutely.

Now, we must check whether $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{4}{2k+1}$converges conditionally.

For this, we use the alternating series test.
We can calculate $\displaystyle\lim_{k\to\infty}\frac{4}{2k+1} = 0$, so that hypothesis is fulfilled.

Also, $0 < \dfrac{4}{2k+3} \leq \dfrac{4}{2k+1}$, because $2k+3 \geq 2k+1$. So, by the AST, this series converges. Therefore, it converges conditionally.

#12. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k}\frac{k^2 3^k}{2^k}$

ANSWER

We will use the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^2 3^{k+1}}{2^{k+1}}\frac{2^k}{k^2 3^k} = \frac{(k+1)^2 3}{2k^2} = \frac{3}{2}\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^2 = \frac{3}{2}1^2 = \frac{3}{2}$$

So,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+1)^2 3}{2k^2} = \frac{3}{2}$$

And $\frac{3}{2} > 1$, therefore the series diverges by the ratio test.

#14. $\sum_{k=1}^{\infty} \left(\frac{1-3k}{4k}\right)^k$

ANSWER

We use the root test:

$$\sqrt[k]{\left|\left(\frac{1-3k}{4k}\right)^k\right|} = \sqrt[k]{\left(\frac{|1-3k|}{4k}\right)^k} = \frac{|1-3k|}{4k} = \frac{3k-1}{4k}$$

So,

$$\lim_{k\to\infty} \sqrt[k]{\left|\left(\frac{1-3k}{4k}\right)^k\right|} = \lim_{k\to\infty} \frac{3k-1}{4k} = \frac{3}{4}$$

Clearly, $\frac{3}{4} < 1$, so this series converges absolutely by the ratio test.

#42. Determine whether $\sum_{k=1}^{\infty} \frac{k!}{1\cdot 3\cdot 5\cdots(2k-1)}$ converges or diverges.

ANSWER

We use the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)!}{1\cdot 3\cdot 5\cdots(2k+1)} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{k!} = \frac{k+1}{2k(2k+1)}$$

So,

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k+1}{2k(2k+1)} = \lim_{k\to\infty} \frac{k+1}{4k^2+2k} = 0$$

Clearly $0 < 1$, so this series converges absolutely by the ratio test.