MATH 231 U1, Spring 2009
Homework 19 (Worksheet) Answers
Due Friday, March 13th, 2009

1. Write the number $1.334\overline{234} = 1.334234234234234234\ldots$ as a ratio of integers.
(Hint: $1.334\overline{234} = 1.1 + \frac{234}{10^3} + \frac{234}{10^6} + \ldots.$)

ANSWER

$$\frac{234}{10^3} + \frac{234}{10^6} + \cdots = \sum_{k=0}^{\infty} \frac{234}{10^3}\left(\frac{1}{10^3}\right)^k$$

and since $\frac{1}{10^3} < 1$ we know this geometric series converges to

$$\frac{a}{1-r} = \frac{\frac{234}{10^3}}{1-10^{-3}} = \frac{234}{999}$$

Now, we add

$$1.1 + \frac{234}{999} = \frac{11}{10} + \frac{234}{999} = \frac{11(999)+2340}{9990} = \frac{13329}{9990}$$

Any repeating decimal may be treated this way, to show it is a rational number, as opposed to an irrational number like π, which is a non-repeating decimal.

3. Use the ratio test to determine the convergence of the following series.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{k \cdot k!}$

ANSWER

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{3^{k+1}}{(k+1)\cdot(k+1)!}\frac{k\cdot k!}{3^k} = \lim_{k\to\infty}\frac{3k}{(k+1)^2} = 0 < 1$$

So, by the ratio test, the series converges absolutely.

b) $\displaystyle\sum_{k=1}^{\infty} k^2\left(\frac{5}{7}\right)^k$

ANSWER

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+1)^2\left(\frac{5}{7}\right)^{k+1}}{(k)^2\left(\frac{5}{7}\right)^k} = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^2\frac{5}{7} = \frac{5}{7} < 1$$

So by the ratio test, the series converges absolutely

c) $\displaystyle\sum_{k=1}^{\infty}\frac{(-4)^k}{k^2}$

ANSWER

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{4^{(k+1)}}{(k+1)^2}\frac{k^2}{4^k} = 4\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^2 = 4\cdot 1^2 = 4 > 1$$

Therefore by the ratio test, the series diverges.

d) $\displaystyle\sum_{k=1}^{\infty}k^p r^k$ for $p > 0$ and $|r| < 1$.

ANSWER

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+1)^p|r^{k+1}|}{k^p|r^k|} = \lim_{k\to\infty}\frac{(k+1)^p|r|}{k^p} = |r|\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^p = |r|\cdot 1^p = |r|$$

Since we are told $|r| < 1$, we know that this series converges absolutely by the ratio test

5. Determine the convergence (abs., cond. or divergent) of the following series.

a) $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{1}{2^k}$

ANSWER

Look at the series of absolute values $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2^k}$. This is a geometric series with $r = \frac{1}{2}$, since $|r| = \frac{1}{2} < 1$ it converges. Therefore the series $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{1}{2^k}$ is absolutely convergent.

b) $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{4k^2}{2k^{5/2}+6k+1}$

ANSWER

First we look at the series of absolute values $\sum_{k=1}^{\infty} \frac{4k^2}{2k^{5/2}+6k+1}$. We will use the limit comparison test to show this series diverges: Let $b_k = \frac{1}{k}$. Then,

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{4k^2}{2k^{5/2}+6k+1}\frac{k}{1} = \lim_{k\to\infty} \frac{4k^3}{2k^{5/2}+6k+1} = \lim_{k\to\infty} \frac{4k^{1/2}}{2+6k^{-3/2}+k^{-5/2}} = \infty$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and since we got ∞ for the limit, we know by the limit comparison test that $\sum_{k=1}^{\infty} \frac{4k^2}{2k^{5/2}+6k+1}$ diverges too (See Prob. 37 or 38 from Section 8.3)

So, our series $\sum_{k=1}^{\infty} (-1)^k \frac{4k^2}{2k^{5/2}+6k+1}$ is not absolutely convergent, but it may still be convergent (and hence it would be *conditionally* convergent. To check this use the alternating series test:

$$\lim_{k\to\infty} \frac{4k^2}{2k^{5/2}+6k+1} = \lim_{k\to\infty} \frac{4k^{-1/2}}{2+6k^{-3/2}+1} = 0$$

so that hypothesis of the AST is true.

Now, we must show that for large enough k, $0 < a_{k+1} \leq a_k$.

Clearly the terms are positive, since the numerators and denominators of the terms will be positive for all $k \geq 1$.

To show the terms are decreasing for large enough k, we will look at the function $f(x) = \frac{4x^2}{2x^{5/2}+6x+1}$, where $f(k) = a_k$, and show it is a decreasing function for large enough x.
This will imply that $0 < a_{k+1} \leq a_k$ for large enough k. This is great CALC I REVIEW!

A function is decreasing wherever its derivative is negative. So we calculate $f'(x)$ and show that it will be negative for large enough x.

$$f'(x) = \frac{(2x^{5/2}+6x+1)8x - 4x^2(5x^{3/2}+6)}{(2x^{5/2}+6x+1)^2} = \frac{16x^{7/2}+48x^2+8x-20x^{7/2}-24x^2}{(2x^{5/2}+6x+1)^2}$$

$$= \frac{-4x^{7/2}+24x^2+8x}{(2x^{5/2}+6x+1)^2}.$$

The denominator here is clearly positive, therefore to find x large enough so that $f'(x)$ is negative, we find x large enough so that $g(x) := -4x^{7/2}+24x^2+8x$ is negative. For example:

$g(100) = -4(10)^7 + 240000 + 800 = -39759200$ is clearly negative.

Then, calculate $g'(x) = -14x^{5/2} + 48x$, and see that $g'(x) = 0$ if and only if $14x^{5/2} = 48x$.

$$\begin{aligned} 14x^{5/2} &= 48x \\ x^{3/2} &= \frac{48}{14} \\ x &= \left(\frac{24}{7}\right)^{2/3} \approx 2.273747 \end{aligned}$$

So, we know that the only place in $[0, \infty)$ that $g'(x)$ can change sign at that x is 2.273747 (This is a CRITICAL POINT of g ...sound familiar?).

We test a point $x = 4 \geq 2.273747$ and see that $g'(4) = -14(4^5) + 48 \cdot 4 = -14144$, which is negative.

This tells us that $g'(x) < 0$ for all $x \geq 2.273747$, and therefore $g(x)$ is decreasing for all $x \geq 2.273747$.

So we know $g(100)$ is negative and since we know $g(x)$ is decreasing for all $x \geq 2.273747$, this guarantees that $g(x)$ will be negative for all $x \geq 100$.
(IT MAY BE NEGATIVE FOR SMALLER x's, but we know for sure that it is negative after 100.)
Therefore $f'(x) = \frac{g(x)}{(2x^{5/2}+6x+1)^2}$ is negative for $x \geq 100$.

So, lets unravel this:
We have just proved $f'(x) < 0$ for $x \geq 100$. Therefore, $f(x)$ is decreasing for $x \geq 100$. This guarantees that $f(k+1) \leq f(k)$ for $k \geq 100$. Therefore, $a_{k+1} \leq a_k$ for $k \geq 100$.

So, by the Alternating Series Test, $\displaystyle\sum_{k=100}^{\infty} (-1)^k \frac{4k^2}{2k^{5/2} + 6k + 1}$ converges. (We showed the other hypothesis of the AST above.)

So,

$$\sum_{k=1}^{\infty} (-1)^k \frac{4k^2}{2k^{5/2} + 6k + 1} = \sum_{k=1}^{99} (-1)^k \frac{4k^2}{2k^{5/2} + 6k + 1} + \sum_{k=100}^{\infty} (-1)^k \frac{4k^2}{2k^{5/2} + 6k + 1}.$$

Which tells us that $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{4k^2}{2k^{5/2} + 6k + 1}$ converges too, because we can write it as a finite sum plus a convergent infinite series.

IMPORTANT NOTE: the exam and quiz questions won't be as long as 5(b)! If you're doing an exam question, and it looks like it will be really really long, you're probably not doing it correctly.

c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k^2}$

ANSWER

Because $1 + k^2 \geq k^2$ we know that $0 \leq \frac{1}{1+k^2} \leq \frac{1}{k^2}$. We also know that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series, since $2 > 1$. So, by the comparison test this series converges too.

d) $\displaystyle\sum_{k=1}^{\infty} \frac{k^7 + 7k^6 + 9k^3 + 14}{15k^5 + 7k^3 + 1}$.

ANSWER

$$\lim_{k\to\infty} \frac{k^7 + 7k^6 + 9k^3 + 14}{15k^5 + 7k^3 + 1} = \lim_{k\to\infty} \frac{k^2 + 7k + 9k^{-2} + 14k^{-5}}{15 + 7k^{-2} + k^{-5}} = \infty$$

So, this series diverges by the kth term test.