MATH 231 U1, Spring 2009
Homework 20 (8.6) Answers
Due Wednesday, April 1st, 2009

#2. Find a power series representation of $f(x) = \dfrac{3}{x-1}$ centered at $c = 0$ and find the radius and interval of convergence. (Graph $f(x)$ together with the partial sums where $n = 3$ and $n = 6$.)

ANSWER

$f(x) = \dfrac{3}{x-1} = \dfrac{-3}{1-x} = \dfrac{a}{1-r}$ when $a = -3$ and $r = x$. So, the power series representation is $\displaystyle\sum_{k=0}^{\infty}(-3)x^k$.

This is a geometric series with ratio x, so it converges if and only if $|x| < 1$, that is, the R.O.C. is 1, and the interval of convergence is $(-1, 1)$. (We need not check the endpoints, for a geometric series they will never be included.)

#14. Determine the radius and interval of convergence for $\displaystyle\sum_{k=0}^{\infty} 3\left(\frac{x}{4}\right)^k$.

ANSWER

This one is a geometric series with ratio $\dfrac{x}{4}$, so it converges when $\left|\dfrac{x}{4}\right| < 1$, that is, when $-4 < x < 4$. It diverges when $\left|\dfrac{x}{4}\right| \geq 1$. So, the radius of convergence is 4 and the interval of convergence is $(-4, 4)$. The series converges to the function $\dfrac{12}{4-x}$. (Alternatively, the ratio test will work to do this problem too.)

#16. Determine the radius and interval of convergence for $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{k!}x^k$.

ANSWER

Here, we will use the ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{3^{k+1}x^{k+1}}{(k+1)!}\frac{k!}{3^k x^k}\right| = \lim_{k\to\infty}\frac{3}{k+1}|x| = 0 < 1$$

So, no matter what x is, the limit of the ratio is 0, which is < 1. So, the series converges absolutely for all x, by the ratio test. Thus, the radius of convergence for the series is ∞ and the interval of convergence is $(-\infty, \infty)$.

#20. Determine the radius and interval of convergence for $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k4^k}(x+2)^k$

ANSWER

Here again, we use the ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+2}(x+2)^{k+1}}{(k+1)4^{k+1}}\frac{k4^k}{(-1)^{k+1}(x+2)^k}\right| = \lim_{k\to\infty}\frac{k}{4(k+1)}|x+2| = \frac{1}{4}|x+2|$$

The quantity $\frac{1}{4}|x+2|$ is < 1 if and only if $|x+2| < 4$, that is, if $-4 < x+2 < 4$, meaning $-6 < x < 2$. So, the radius of convergence is 4. To determine the interval of convergence, we must check to see if $x = -6$ and $x = 2$ are included in the interval or convergence or not.

When $x = 2$, the power series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k4^k}(4)^k = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$$

which converges by the AST.

When $x = -6$, the power series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k4^k}(-4)^k = \sum_{k=1}^{\infty}\frac{(-1)^{2k+1}}{k} = \sum_{k=1}^{\infty}\frac{-1}{k} = -\sum_{k=1}^{\infty}\frac{1}{k}$$

which diverges, because it is a constant times the harmonic series.

So, the I.O.C. is $(-6, 2]$.

#34. Find the power series representation and radius of convergence for $f(x) = \dfrac{3}{(x-1)^2}$ by integrating or differentiating one of the series from problems 1-8.

ANSWER

Look at the function $g(x) = \dfrac{3}{x-1}$ from Problem 2. The function $f(x)$ above is $-g'(x)$. We know from Problem 2 that $g(x) = \sum_{k=0}^{\infty}(-3)x^k$ on the interval of convergence $(-1, 1)$. So, we take the derivative of the power series and get

$$f(x) = -g'(x) = -\sum_{k=1}^{\infty}(-3)kx^{k-1} = \sum_{k=1}^{\infty}3kx^{k-1}$$

The R.O.C. of this series is the same as the R.O.C. of the series for $g(x)$, which was 1.

#44. If $\sum_{k=0}^{\infty}a_kx^k$ has radius of convergence R, with $0 < R < \infty$, determine the radius of convergence of $\sum_{k=0}^{\infty}a_k\left(\frac{x}{b}\right)^k$ for any constant $b \neq 0$.

ANSWER

The center of this power series $\sum_{k=0}^{\infty} a_k x^k$ is 0, and it converges on $(-R, R)$, that is, it converges if $|x| < R$. We don't know whether it converges at the endpoints R and $-R$, but it does not matter for this question. The series must diverge for $|x| > R$ (otherwise, the R.O.C. would be larger than R.)

Therefore, $\sum_{k=0}^{\infty} a_k \left(\frac{x}{b}\right)^k$ converges whenever $\left|\frac{x}{b}\right| < R$ (and it diverges whenever $\left|\frac{x}{b}\right| > R$.) The inequality $\left|\frac{x}{b}\right| < R$ is true if and only if $|x| < R|b|$. Thus the R.O.C. for $\sum_{k=0}^{\infty} a_k \left(\frac{x}{b}\right)^k$ is $R|b|$.