MATH 231 U1, Spring 2009
Homework 21 (8.7 part 1) Answers
Due Friday, April 3rd, 2009

#2. Find the Maclaurin series (i.e. Taylor series at c=0) and its interval of convergence for $f(x) = \sin x$.

ANSWER

Using the formula for Taylor series, we know that the series has form $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. (Recall that $0! = 1$.)

So, we must find the coefficients $\frac{f^{(k)}(0)}{k!}$ for $f(x) = \sin x$.

$$
\begin{aligned}
f(x) &= \sin x \\
f'(x) &= \cos x \\
f''(x) &= -\sin x \\
f^{(3)}(x) &= -\cos x \\
f^{(4)}(x) &= \sin x
\end{aligned}
$$

So, plugging in 0 we get

$$
\begin{aligned}
f(0) &= \sin 0 = 0 \\
f'(0) &= \cos 0 = 1 \\
f''(0) &= -\sin 0 = 0 \\
f^{(3)}(0) &= -\cos 0 = -1 \\
f^{(4)}(0) &= \sin 0 = 0
\end{aligned}
$$

These derivatives repeat in cycle of 4, and every even coefficient is 0. So, we can write the Maclaurin series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + ... =$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

To find the interval and radius of convergence, use the Ratio test:

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{x^{2k+3}}{(2k+3)!} \frac{(2k+1)!}{x^{2k+1}} \right| = \lim_{k \to \infty} \frac{x^2}{(2k+3)(2k+2)} = 0$$

Which means the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is ∞.

#16. Graph $f(x)$ and the Taylor polynomials for the indicated center c and degree n. $f(x) = \dfrac{1}{1+x}$ $c = 0$, $n = 4$; $n = 8$.

ANSWER

There are several ways to approach this problem. The shorter one is by using the fact that $\frac{1}{1+x}$ can be written as the sum of a geometric series (since it is easily written as $\frac{a}{1-r}$ for $a = 1$ and $r = -x$.)

So, we know that

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}(-1)^k x^k = 1 - x + x^2 - x^3 + \ldots.$$

Which converges if and only if $|-x| < 1$, that is, it converges on $(-1, 1)$.

This power series representation of $f(x)$ IS THE TAYLOR SERIES for $f(x)$ centered at c. This is because there is at most ONE way to represent any given function as a power series with a given center c. So, if there is an easier way to find the power series representation for a given function at a given center c, you don't have to use formula for the coefficients of the Taylor series.

Now that we have found the Taylor series, the Taylor polynomial P_4 is just the first 5 terms of the Taylor series, out to the x^4 term, so

$$P_4(x) = 1 - x + x^2 - x^3 + x^4$$

and similarly

$$P_8(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8.$$

You should graph $f(x)$ and $P_4(x)$ and $P_8(x)$ to observe the convergence!

The other way to do this problem is by using the formula for the coefficients of the Taylor series. The Taylor polynomial P_4 of degree 4 centered at 0 is defined as:

$$P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

And

$$P_8(x) = \sum_{k=0}^{8} \frac{f^{(k)}(0)}{k!}x^k$$

So, to find the coefficients for these Taylor series, we need to find a bunch of derivatives of f.

$$
\begin{aligned}
f(x) &= \frac{1}{1+x} = (1+x)^{-1} \\
f'(x) &= -(1+x)^{-2} \\
f''(x) &= 2(1+x)^{-3} \\
f^{(3)}(x) &= -3!(1+x)^{-4} \\
f^{(4)}(x) &= 4!(1+x)^{-5} \\
f^{(5)}(x) &= -5!(1+x)^{-6} \\
f^{(6)}(x) &= 6!(1+x)^{-7} \\
f^{(7)}(x) &= -7!(1+x)^{-8} \\
f^{(8)}(x) &= 8!(1+x)^{-9}
\end{aligned}
$$

And take their values at our center $c = 0$:

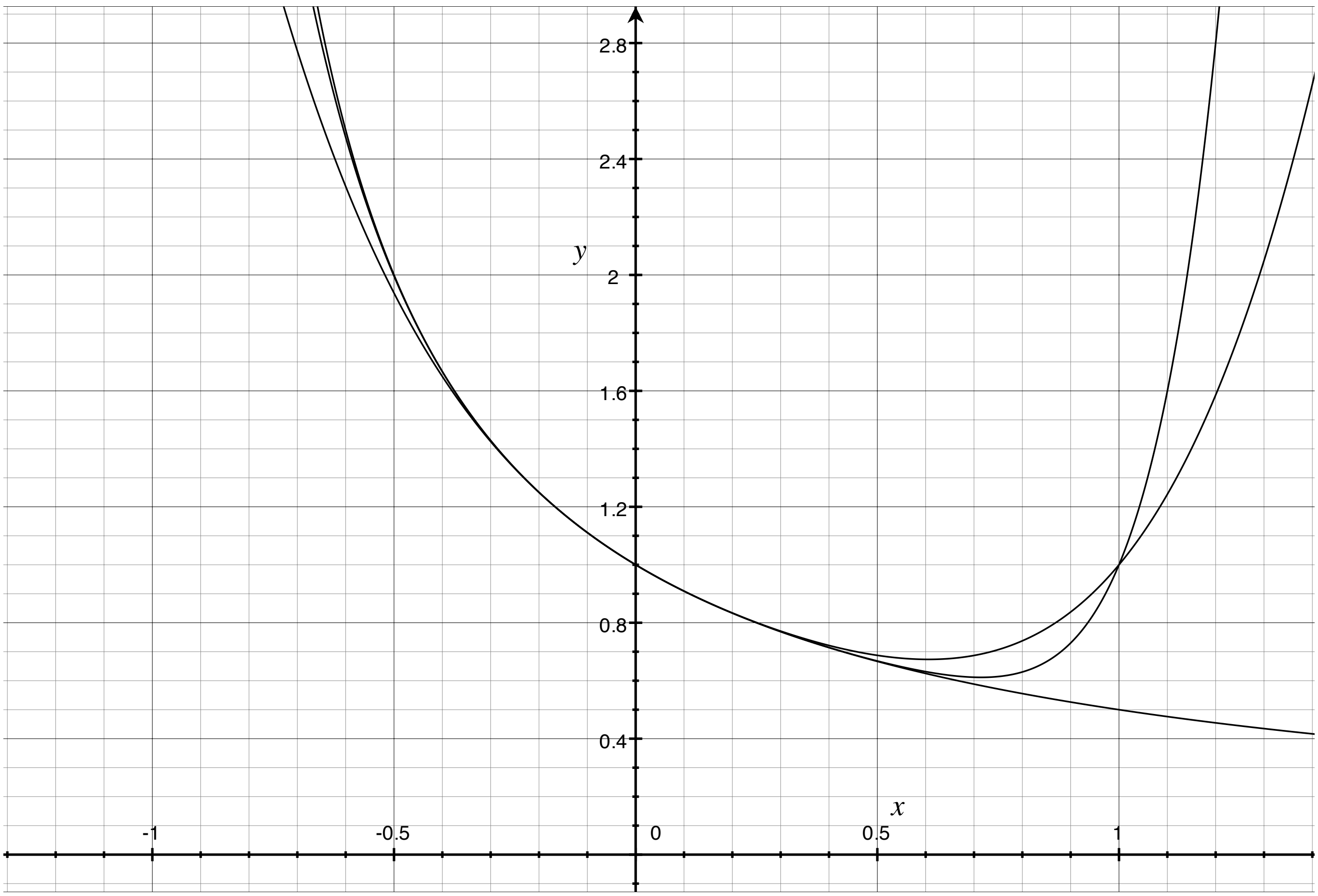
$$
\begin{aligned}
f(0) &= \frac{1}{1+0} = 1 \\
f'(0) &= -(1+0)^{-2} = -1 \\
f''(0) &= 2 \\
f^{(3)}(0) &= -3! \\
f^{(4)}(0) &= 4! \\
f^{(5)}(0) &= -5! \\
f^{(6)}(0) &= 6! \\
f^{(7)}(0) &= -7! \\
f^{(8)}(0) &= 8!
\end{aligned}
$$

Notice that these numbers have the pattern $f^{(k)}(0) = (-1)^k k!$, so, this means our coefficients will be $\dfrac{f^{(k)}(0)}{k!} = \dfrac{(-1)^k k!}{k!} = (-1)^k$. So,

$$P_4(x) = 1 - x + x^2 - x^3 + x^4$$

and

$$P_8(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8$$



#24. Prove that the Taylor series converges to $f(x)$ by showing that $R_n(x) \to 0$ as $n \to \infty$.

$$e^{-x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$$

ANSWER

By Taylor's Theorem, we know that for each n the remainder term

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1} = \frac{(-1)^{n+1}e^{-z}}{(n+1)!}x^{n+1}$$

for some z strictly between x and 0 (and z also depends on which n we are looking at.)

In order to show $R_n(x) \to 0$ as $n \to \infty$, it is enough to show that $|R_n(x)| \to 0$ as $n \to \infty$. (This is a fact from the sequences section 8.1)

So, we want to show

$$|R_n(x)| = \frac{e^{-z}}{(n+1)!}|x^{n+1}|$$

has limit 0 as $n \to \infty$.

First, we find a bound on the size of e^{-z}.
Case I:
If $x > 0$, then the fact that z is between 0 and x means $z > 0$. Therefore, $e^{-z} < 1$.
Case II:
If $x < 0$, then $z < 0$, and we know that $e^{-z} = e^{|z|} < e^{|x|} = e^{-x}$.

So, if we let

$$M = \max\{1, e^{-x}\}$$

then for any x and any n we know $e^{-z} \leq M$.

Therefore,

$$|R_n(x)| = \frac{e^{-z}}{(n+1)!}|x^{n+1}| \leq M\frac{|x|^{n+1}}{(n+1)!}$$

So,

$$0 \leq \lim_{n\to\infty} |R_n(x)| \leq M \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}$$

And you can show $\displaystyle\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ just as they do at the end of Example 7.3.

Therefore $\lim_{n\to\infty} |R_n(x)| = 0$.