MATH 231 U1, Spring 2009
Homework 22 (8.7 part 2) Answers
Due Monday, April 5th, 2009

#14. Find the Taylor series for $f(x) = \frac{1}{x}$ centered at $c = -1$ and determine its interval of convergence.

ANSWER

THE LONG WAY:

$$\begin{aligned} f(x) &= \frac{1}{x} \\ f'(x) &= -x^{-2} \\ f''(x) &= 2x^{-3} \\ f^{(3)}(x) &= -2 \cdot 3x^{-4} \\ f^{(4)}(x) &= 2 \cdot 3 \cdot 4x^{-5} \\ f^{(5)}(x) &= -5!x^{-6} \end{aligned}$$

and we plug in $c = -1$

$$\begin{aligned} f(-1) &= -1 \\ f'(1) &= -1 \\ f''(x) &= -2 \\ f^{(3)}(x) &= -3! \\ f^{(4)}(x) &= -4! \\ f^{(5)}(x) &= -5! \end{aligned}$$

So, for $k \geq 0$

$$f^{(k)}(-1) = -k!$$

and

$$\frac{f^{(k)}(-1)}{k!} = -1$$

and this makes the Taylor series $\sum_{k=0}^{\infty} -(x+1)^k$.

THE SHORTER WAY:
Note that $\frac{1}{x} = \frac{-1}{1-(x+1)}$. Thus, $\frac{1}{x} = \sum_{k=0}^{\infty} -(x+1)^k$. Since this is a geometric series, it converges if and only if $|x+1| < 1$, that is when $-2 < x < 0$. So the I.O.C. is $(-2, 0)$.

#28. (a) Use a Taylor Polynomial with degree 4 to approximate $\sqrt{1.2}$
(b) Estimate the error in your approximation.
(c) Estimate the number of terms needed in a Taylor polynomial to guarantee an accuracy of 10^{-10}

ANSWER

(a)
We will use the Taylor polynomial $P_4(x)$ for $\sqrt{x}$ centered at 1, since 1 is close to 1.2.

$$\begin{aligned}
f(x) &= \sqrt{x} = x^{1/2} \\
f'(x) &= \frac{1}{2}x^{-1/2} \\
f''(x) &= -\frac{1}{4}x^{-3/2} \\
f^{(3)}(x) &= \frac{3}{8}x^{-5/2} \\
f^{(4)}(x) &= -\frac{15}{16}x^{-7/2} \\
f^{(5)}(x) &= \frac{15\cdot 7}{32}x^{-9/2}
\end{aligned}$$

plug in 1

$$\begin{aligned}
f(1) &= 1 \\
f'(1) &= \frac{1}{2} \\
f''(1) &= -\frac{1}{4} \\
f^{(3)}(1) &= \frac{3}{8} \\
f^{(4)}(1) &= -\frac{15}{16} \\
f^{(5)}(1) &= \frac{15\cdot 7}{32}
\end{aligned}$$

So,

$$P_4(x) = 1 + \frac{1}{2}x - \frac{1}{4\cdot 2!}x^2 + \frac{3}{8\cdot 3!}x^3 - \frac{15}{16\cdot 4!}x^4$$

and if we calculate $P_4(1.2) = 1.0954$, this is our estimate of $\sqrt{1.2}$.

(b) We will use Taylor's Theorem to estimate the error

$$R_4(1.2) = \frac{f^{(5)}(z)}{5!}(1.2-1)^5$$

for some z between 1 and 1.2.

We know $f^{(5)}(x) = \dfrac{15 \cdot 7}{32} x^{-9/2}$, and this is a decreasing function on $(1, 1.2)$. So, $f^{(5)}(1) \geq f^{(5)}(z)$ for all $z \in (1, 1.2)$, which means $f^{(5)}(z) \leq f^{(5)}(1) = \frac{15 \cdot 7}{32}$.

So,

$$R_4(1.2) = \frac{f^{(5)}(z)}{5!}(1.2 - 1)^5 \leq \frac{15 \cdot 7}{32 \cdot 5!}(0.2)^5$$

(c) We need to estimate $|R_n(1.2)| = |\sqrt{1.2} - P_n(1.2)|$.

We know

$$R_n(1.2) = \frac{f^{(n+1)}(z)}{(n+1)!}(1.2 - 1)^{n+1}$$

by Taylor's theorem.

By observing the pattern in our calculations from part (a) we see that for $k \geq 2$

$$f^{(k)}(x) = (-1)^k \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2k-3)}{2^k} x^{(2k-1)/2}$$

And, for any k,

$$|f^{(k)}(x)| = \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2k-3)}{2^k} |x^{(2k-1)/2}|$$

is a decreasing function on $(1, 1.2)$, meaning that for all $z \in (1, 1.2)$

$$|f^{(k)}(1)| = \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2k-3)}{2^k} \geq |f^{(k)}(z)|.$$

Therefore,

$$|R_n(1.2)| = |\frac{f^{(n+1)}(z)}{(n+1)!}(1.2 - 1)^{n+1}| \leq \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2(n+1) - 3)}{2^{n+1}(n+1)!}(0.2)^{n+1}.$$

By guess and check, we see that any $n \geq 12$ makes this quantity $< 10^{-10}$. (This may not be the *best* n, but it works.)

#38. Use a known Taylor series to find the Taylor series about $c = 0$ for the given function and find its radius of convergence.
$f(x) = \cos x^3$.

The Taylor series for $\cos x$ centered at 0 is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

which has I.O.C. $(-\infty, \infty)$.

Plug in x^3 to get

$$\cos x^3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k} = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + ...$$

for $x^3 \in (-\infty, \infty)$, that is, for $x \in (-\infty, \infty)$, which means the I.O.C. of this new series is also $(-\infty, \infty)$.

#43. Prove that if f and g are functions such that $f''(x)$ and $g''(x)$ exist and are continuous for all x and

$$\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^2} = 0$$

then $f(a) = g(a)$, $f'(a) = g'(a)$ and $f''(a) = g''(a)$. What does this imply about the Taylor series?

ANSWER

The fact that f'' and g'' exist means that f' and g' are continuous and f and g are continuous.

We assumed that $\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^2} = 0$ and we may clearly calculate $\lim_{x \to a} (x-a)^2 = 0$, which implies that $\lim_{x \to a} (f(x) - g(x)) = 0$ (since otherwise, $\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^2}$ couldn't be 0.)

So we know $\lim_{x \to a} (f(x) - g(x)) = 0$, which implies $f(a) = g(a)$ because f and g are continuous.

Next, because $\lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^2}$ has indeterminate form $\frac{0}{0}$, we may use L'Hopital's rule to show:

$$0 = \lim_{x \to a} \frac{f(x) - g(x)}{(x-a)^2} = \lim_{x \to a} \frac{f'(x) - g'(x)}{2(x-a)}$$

and therefore

$$\lim_{x \to a} \frac{f'(x) - g'(x)}{2(x-a)} = 0.$$

Now, since the limit of the denominator $2(x-a)$ is 0 as $x \to a$, in order for this limit to be 0, the limit of the numerator must also be 0. Therefore $\lim_{x \to a} (f'(x) - g'(x)) = 0$. Then, the fact that f' and g' are continuous implies that $f'(a) = g'(a)$.

Next, because $\lim_{x \to a} \frac{f'(x) - g'(x)}{(2(x-a)}$ has indeterminate form $\frac{0}{0}$, we may use L'Hopital's rule to show:

$$0 = \lim_{x \to a} \frac{f'(x) - g'(x)}{2(x-a)} = \lim_{x \to a} \frac{f''(x) - g''(x)}{2}$$

and therefore

$$\lim_{x \to a} \frac{f''(x) - g''(x)}{2} = 0.$$

Which clearly means that $\lim_{x \to a} (f''(x) - g''(x)) = 0$, and then so long as f'' is continuous, this implies $f''(a) = g''(a)$.

#44. Generalize #43 by showing if f and g are functions such that $f^{(n)}(x)$ and $g^{(n)}(x)$ exist for all x and

$$\lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^n}=0$$

then $f(a)=g(a)$, $f'(a)=g'(a)...f^{(n)}(a)=g^{(n)}(a)$.

ANSWER

The proof here goes exactly as in #43, use L'Hopital's rule n times.