MATH 231 U1, Spring 2009
Homework 23 (8.8) Answers
Due Wednesday, April 7th, 2009

#5. Use an appropriate Taylor series to approximate $e^{-0.2}$, accurate to within 10^{-11}.

ANSWER

The Taylor series for e^x centered at 0 is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and it has I.O.C. $(-\infty, \infty)$. This is the series we will use. Recall that the Taylor Polynomial of degree n is $P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

By Taylor's Theorem, for $x = -0.2$ the error in using $P_n(-0.2)$ to approximate $f(-0.2) = e^{-0.2}$

$$|R_n(-0.2)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(-0.2)^{n+1} \right|$$

for some $z \in (-0.2, 0)$.

And $f^{(n+1)}(x) = e^x$, so, for $z \in (-0.2, 0)$, we know $|f^{(n+1)}(z)| = e^z \leq 1$. Therefore,

$$|R_n(-0.2)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(-0.2)^{n+1} \right| \leq \frac{1}{(n+1)!}|(-0.2)^{n+1}|$$

Now, by guess and check can find n so that,

$$\frac{1}{(n+1)!}|(-0.2)^{n+1}| < 10^{-11}.$$

Then, plug -0.2 in to $P_n(x)$ for that n. That number will be your approximation.

#10. Use a known Taylor series to conjecture the value of the limit for

$$\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3}$$

The Taylor series for $\tan^{-1} x$ centered at 0 is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + ...$$

and it has I.O.C $[-1, 1]$, so it certainly converges for x near 0. So, we will use this power series representation of $\tan^{-1} x$.

$$\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3} = \lim_{x\to 0} \frac{(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + ...) - x}{x^3} = \lim_{x\to 0} \frac{(-\frac{1}{3}x^3 + \frac{1}{5}x^5 + ...)}{x^3} = \lim_{x\to 0} (\frac{-1}{3} + \frac{1}{5}x^2 + ...) = \frac{-1}{3}$$

#14. Use a known Taylor polynomial with $n = 4$ non-zero terms to estimate $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos x^2 \, dx$.

The Taylor series for $\cos x$ centered at 0 is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

and it converges on $(-\infty, \infty)$.

So, the Taylor series for $\cos x^2$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + ...$$

and it also converges on $(-\infty, \infty)$.

Therefore, the Taylor polynomial with 4 nonzero terms is $P_{12}(x) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!}$

And

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos x^2 \, dx \approx \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \left(1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!}\right) dx = \left[x - \frac{x^5}{10} + \frac{x^9}{4! \cdot 9} - \frac{x^{13}}{6! \cdot 13}\right]_{-\sqrt{\pi}}^{\sqrt{\pi}}$$

which you can easily calculate.