MATH 231 U1, Spring 2009
Homework 24 (worksheet) Answers
Due Friday, April 10th, 2009

#2. What is the Taylor series for $x^5 - 3x^4 + 100x - 2$ centered at $x = 0$?

ANSWER

It is its own Taylor series.

#3. Recall $\int \ln x\, dx = x \ln x - x + C$. Find the Taylor Series for $x \ln x - x$ centered at 1 by

(b) Integrating the series for $\ln x$

ANSWER

$$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$$\int \ln x\ dx = \int \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$$x \ln x - x + C = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{(x-1)^{k+1}}{k+1}$$

To find C, plug in $x = 1$, which makes most of the terms in the power series 0.

$$1 \ln 1 - 1 + C = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{(x-x)^{k+1}}{k+1} = 0$$

$$-1 + C = 0$$

So $C = 1$ and

$$x \ln x - x = -1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)}(x-1)^{k+1}$$

#4. (c) Where would be a better place to center your series to approximate $\sin(3.15)$?

ANSWER

It would be better to center the series at $c = \pi$, this Taylor series will converge faster than the one centered at 0 for $x = 3.15$, so you will have less error in the approximation

(d) Find the Taylor Series for $\sin x$ at that center from part (c), and use Taylor's Theorem to estimate the error in using the first 4 nonzero terms of *this* Taylor series to estimate sin(3.15).

ANSWER

The Taylor series centered at π is $\sum_{k=0}^{\infty} b_k(x-\pi)^k$ where $b_k = \dfrac{f^{(k)}(\pi)}{k!}$. So it is:

$$f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f^{(3)}(\pi)}{3!}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x-\pi)^4 + \frac{f^{(5)}(\pi)}{5!}(x-\pi)^5 + ...$$

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

evaluate these at the center to find the coefficients $b_k = \dfrac{f^{(k)}(\pi)}{k!}$

$$\begin{aligned} f(\pi) &= \sin \pi = 0 \\ f'(\pi) &= \cos \pi = -1 \\ f''(\pi) &= -\sin \pi = 0 \\ f^{(3)}(\pi) &= -\cos \pi = 1 \\ f^{(4)}(\pi) &= \sin \pi = 0 \end{aligned}$$

So, the Taylor series is

$$0 - (x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3 + 0 - \frac{1}{5!}(x-\pi)^5 + 0 + \frac{1}{7!}(x-\pi)^7 ...$$

which, in sigma notation is

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}(x-\pi)^{2k+1}$$

The first 4 nonzero terms of the the series give the Taylor Polynomial of degree 7, $P_7(x)$. So, Taylor's Theorem tells us how to estimate the error:

$$|R_7(3.15)| = \left| \frac{f^{(8)}(z)}{8!}(3.15-\pi)^8 \right|$$

for some z in $(\pi, 3.15)$.

We know that

$$|f^{(8)}(z)| \leq 1$$

because $f^{(8)} = \sin x$. We also know that $3.15 - \pi < 0.01$.

So

$$|R_7(3.15)| = \left| \frac{f^{(8)}(z)}{8!}(x-\pi)^8 \right| \leq \frac{1}{8!}(3.15-\pi)^8 < \frac{1}{8!}(0.01)^8$$

#5. Write the following Power series in sigma notation, and find the interval of convergence.
(a) $5 + 5x^2 + 5x^4 + 5x^6 + 5x^8 + ...$
(also, what function does this series converge to?)

ANSWER

This is a geometric series with ratio x^2 and first term 5, so, it is

$$\sum_{k=0}^{\infty} 5x^{2k}$$

and it converges if and only if $|x^2| < 1$, that is, for $-1 < x < 1$. So, the I.O.C. is $(-1, 1)$. It converges to $\dfrac{5}{1-x^2}$.

(Before we corrected the typo in class, the series was $5 + 5x^{2k} + 5x^{4k} + 5x^{6k} + 5x^{8k} + ...$
we could still answer that question, our answer would just have a k in it.)