MATH 231 U1, Spring 2009
ANSWERS to HW 25 (Section 8.8 # 36, plus supplemental worksheet)
Due Monday, April 13th, 2009

Section 8.8 # 36. Use the Binomal Theorem to find the first 5 terms of the Maclaurin series of $(1+x^2)^{4/5}$.

ANSWER

We start by finding the first 5 terms of the series for $(1+x)^{4/5}$: The Binomal series for $r = \frac{4}{5}$ says:

$$(1+x)^{4/5} = \sum_{k=0}^{\infty} \binom{4/5}{k} x^k$$

for $x \in (-1,1)$.

We find the coefficients using the formula:

$$\binom{4/5}{0} = 1$$
$$\binom{4/5}{1} = 4/5$$
$$\binom{4/5}{2} = \frac{r(r-1)}{2!} = -\frac{\frac{4}{5}\frac{1}{5}}{2!} = -\frac{2}{25}$$
$$\binom{4/5}{3} = \frac{r(r-1)(r-2)}{3!} = \frac{\frac{4}{5}\frac{1}{5}\frac{6}{5}}{6} = \frac{4}{125}$$
$$\binom{4/5}{4} = \frac{r(r-1)(r-2)(r-3)}{4!} = -\frac{\frac{4}{5}\frac{1}{5}\frac{6}{5}\frac{11}{5}}{24} = -\frac{11}{625}$$

So, we get

$$(1+x)^{4/5} = 1 + \frac{4}{5}x - \frac{2}{25}x^2 + \frac{4}{125}x^3 - \frac{11}{625}x^4 + ...$$

Then, to get the terms of the Maclaurin series for $(1+x^2)^{4/5}$, we plug x^2 into the series for $(1+x)^{4/5}$.

So, we get

$$(1+x^2)^{4/5} = 1 + \frac{4}{5}x^2 - \frac{2}{25}x^4 + \frac{4}{125}x^6 - \frac{11}{625}x^8 + ...$$

1. Say the population of Tribbles on the Starship Enterprise is given by the function $f(t)$ where t is in hours. Say you know the rate of change of the population is:

$$f'(t) = te^{t^3}$$

and the ship starts off at time $t = 0$ hours with 2 Tribbles. Use the first 3 non-zero terms of the Taylor series for $f'(t)$ to estimate the population of Tribbles on the Enterprise after 5 hours. (Do you remember how to find the net change in a function $f(x)$ over an interval $[a, b]$ if all you know is $f'(x)$? See Section 4.5, especially Examples 5.10 and 5.11 for a review of the necessary Calc I concepts here!)

ANSWER

The net change in the population $f(t)$ of Tribbles over the time interval $[0, 5]$ is given by

$$\int_0^5 f'(t)\, dt = \int_0^5 te^{t^3}\, dt$$

We will estimate the value of this integral by integrating the first 3 non-zero terms of the Taylor series for e^{t^3}

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + ...$$

$$e^{t^3} = \sum_{k=0}^{\infty} \frac{t^{3k}}{k!} = 1 + t^3 + \frac{1}{2}t^6 + \frac{1}{3!}t^9 + ...$$

$$te^{t^3} = \sum_{k=0}^{\infty} \frac{t^{3k+1}}{k!} = t + t^4 + \frac{1}{2}t^7 + \frac{1}{3!}t^{10} + ...$$

$$\int_0^5 te^{t^3}\, dt \approx \int_0^5 \left(t + t^4 + \frac{1}{2}t^7\right) dt = \frac{1}{2}t^2 + \frac{1}{5}t^5 + \frac{1}{16}t^8\Big|_0^5 = 25051.5625$$

To get the final answer, since we started with 2 Tribbles, we add the estimated net change to the original value to get

$$f(5) \approx 2 + 25051.5625 = 25053.5625 \approx 25054 \text{ Tribbles.}$$

2. (a) Find the Taylor series for $g(x) = (x-1)^2 \ln x$ centered at 1 by manipulating the series you memorized for $\ln x$.

ANSWER

$$(x-1)^2 \ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^{k+2} = (x-1)^3 - \frac{1}{2}(x-1)^4 + \frac{1}{3}(x-1)^5 - \frac{1}{4}(x-1)^6 + ...$$

(b) Use the Taylor series above to find the values of $g''(1)$, $g^{(20)}(1)$ and $g^{(97)}(1)$.

The Taylor series for $g(x)$ centered at 1 is $\sum_{k=0}^{\infty} b_k(x-1)^k$, where the coefficient b_k of the $(x-1)^k$ term is $b_k = \dfrac{g^{(k)}(1)}{k!}$.

The coefficient of the $(x-1)^2$ term in the Taylor series from part (a) is 0, so by the formula for the Taylor series we know

$$0 = \frac{g''(1)}{2!}$$

and therefore $g''(1) = 0$.

The coefficient of the $(x-1)^{(20)}$ term in the Taylor series from part (a) will be $-\dfrac{1}{18}$, so by the formula for the Taylor series we know

$$-\frac{1}{18} = \frac{g^{(20)}(1)}{20!}$$

and therefore $g^{(20)}(1) = -\frac{20!}{18}$

The coefficient of the $(x-1)^{(97)}$ term in the Taylor series from part (a) will be $\dfrac{1}{95}$, so by the formula for the Taylor series we know

$$\frac{1}{95} = \frac{g^{(97)}(1)}{97!}$$

and therefore $g^{(97)}(1) = \frac{97!}{95}$