

MATH 231 U1, Spring 2009
Answers to HW 28 (section 9.2)
Due Monday, April 20th, 2009

#2. Find the slopes of the tangent lines to the given curves at the indicated points.

$$x = t^3 - t \quad y = t^4 - 5t^2 + 4$$

(a) $t = -1$ (b) $t = 1$ (c) $(0, 4)$.

ANSWER

We calculate $x'(t) = 3t^2 - 1$ and $y'(t) = 4t^3 - 10t$. Then,

$$\left. \frac{dy}{dx} \right|_t = \frac{y'(t)}{x'(t)} = \frac{4t^3 - 10t}{3t^2 - 1}.$$

(a) When $t = -1$ this formula gives us

$$\left. \frac{dy}{dx} \right|_t = \frac{y'(t)}{x'(t)} = \frac{-4 + 10}{3 - 1} = 3.$$

(b) When $t = 1$ we get

$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{y'(1)}{x'(1)} = \frac{4 - 10}{3 - 1} = -\frac{6}{2} = -3.$$

(c) Solve the system $x(t) = 0$, $y(t) = 4$ to see that the point $(0, 4)$ corresponds to $t = 0$.

$$\left. \frac{dy}{dx} \right|_{t=0} = \frac{y'(0)}{x'(0)} = \frac{0}{-1} = 0.$$

#10. Identify all points where the curve has (a) a horizontal tangent and (b) a vertical tangent.

$$x = \cos 2t \quad y = \sin 7t$$

ANSWER

(a) We must determine all of the points where $\frac{dy}{dx} = 0$. To start with, we find the points where $y'(t) = 0$.

$y'(t) = 7 \cos 7t$, which is equal to 0 precisely when $7t = \frac{\pi}{2} + \pi k$ where k is an integer, that is, $y'(t) = 0$ for

$$t = \frac{\pi}{14} + \frac{\pi}{7}k = \frac{\pi + 2k\pi}{14} = \frac{(2k + 1)\pi}{14}$$

where k is an integer.

We must also watch out that at these points, $x'(t) = -2 \sin 2t \neq 0$, since in the case that $y'(c) = x'(c) = 0$, we must take the limit $\lim_{t \rightarrow c} \frac{y'(t)}{x'(t)}$ to find $\frac{dy}{dx}$.

$x'(t) = -2 \sin 2t$ is equal to 0 when $2t = \pi k$ where k is an integer, that is, when $t = \frac{\pi}{2}k$ where k is an integer.

So, if $\frac{(2k+1)\pi}{14}$ is a multiple of $\frac{\pi}{2}$, then it will be a point where $x'(t)$ and $y'(t)$ are both 0. This happens exactly when $(2k+1)$ is a multiple of 7.

So, there are horizontal tangent lines at:

$$t = \frac{\pi}{14}, \text{ which corresponds to the point } (0.90096, 1)$$

$$t = \frac{3\pi}{14} \text{ which corresponds to the point } (0.22252, -1)$$

$$t = \frac{5\pi}{14}, \text{ which corresponds to the point } (-0.6235, 1)$$

$$t = \frac{9\pi}{14}, \text{ which corresponds to the point } (-0.6235, 1) \text{ (the same as } t = \frac{5\pi}{14})$$

$$t = \frac{11\pi}{14}, \text{ which corresponds to the point } (0.22252, -1) \text{ (the same as } t = \frac{3\pi}{14})$$

$$t = \frac{13\pi}{14}, \text{ which corresponds to the point } (0.90096, 1) \text{ (the same as } t = \frac{\pi}{14})$$

$$t = \frac{15\pi}{14}, \text{ which corresponds to the point } (0.90096, -1)$$

$$t = \frac{17\pi}{14}, \text{ which corresponds to the point } (0.22252, 1)$$

$$t = \frac{19\pi}{14}, \text{ which corresponds to the point } (0.6235, -1)$$

and all the rest give you the same 6 points.

When $(2k+1)$ is a multiple of 7, we look at

$$\lim_{t \rightarrow \frac{(2k+1)\pi}{14}} \frac{y'(t)}{x'(t)}$$

by L'Hopital's rule we can see this is equal to either $\frac{49}{4}$ if k is even and is equal to $-\frac{49}{4}$ if k is odd.

(b) Now we are interested in when $x'(t) = 0$ and $y'(t) \neq 0$ in order to find when $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$ is undefined. We have already discussed all the points where both $x'(t)$ and $y'(t) = 0$ and seen that they do not have vertical tangent lines.

We already said $x'(t) = 0$ if and only if $t = \frac{\pi k}{2}$ for any integer k , and we know to rule out the

numbers of the form $\frac{(2k+1)\pi}{14}$, that is, we skip any odd multiples of π for $k \geq 3$. So, there is a vertical tangent at:

$t = 0$, which corresponds to the point $(1, 0)$

$t = \pi$, which corresponds to the point $(1, 0)$

$t = \frac{3\pi}{2}$ can be written as $\frac{21\pi}{14}$, so we skip it...

and all the rest correspond to the point $(1, 0)$.

#18. Given the parametric equations

$$x = 40t + 5 \quad y = 20 + 3t - 16t^2$$

for the position of an object, find the object's velocity and speed at the given times, and describe its motion.

(a) $t = 0$

(b) $t = 2$

ANSWER

We calculate $x'(t) = 40$ and $y'(t) = 3 - 32t$

(a) So at time $t = 0$ the horizontal velocity is $x'(0) = 40$ and the vertical velocity is $y'(0) = 3$, which means the object is moving mostly to the right and a little bit up at a speed of $\sqrt{40^2 + 9} = \sqrt{1609}$.

(b) At time $t = 2$, the horizontal velocity is $x'(2) = 40$ and the vertical velocity is $y'(2) = -61$ which means the object is moving down and to the right with a speed of $\sqrt{40^2 + 61^2}$.

#26. Find the area enclosed by the given curve.

$$x = t \sin t \quad y = t \cos t$$

for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

ANSWER

By plotting some points to get a rough sketch of the curve, you can see that this curve is traced out clockwise. So we use the formula

$$A = \int_c^d y(t)x'(t) dt$$

to calculate the area inside the curve.

$$x'(t) = \sin t + t \cos t$$

So,

$$A = \int_{-\pi/2}^{\pi/2} t \cos t (\sin t + t \cos t) dt = \int_{-\pi/2}^{\pi/2} (t \cos t \sin t + t^2 \cos^2 t) dt$$

and here is where you must dust off your integration skills. (good practice for the final exam!)

First, use trig identities to get

$$\int_{-\pi/2}^{\pi/2} (t \cos t \sin t + t^2 \cos^2 t) dt = \frac{1}{2} \int_{-\pi/2}^{\pi/2} t \sin(2t) dt + \int_{-\pi/2}^{\pi/2} \frac{t^2}{2} (1 + \cos(2t)) dt$$

We'll do $\frac{1}{2} \int_{-\pi/2}^{\pi/2} t \sin(2t) dt$ first:

We use integration by parts (IBP) with $u = t$ and $dv = \sin 2t$. This gives us $du = dt$ and $v = \frac{1}{2} \cos 2t$, and by the IBP formula

$$\frac{1}{2} \int t \sin(2t) dt = \frac{1}{2} \left[\frac{t}{2} \cos 2t - \frac{1}{2} \int \cos 2t dt \right] = \frac{1}{2} \left[\frac{t}{2} \cos 2t + \frac{1}{4} \sin 2t \right] = \frac{t}{4} \cos 2t + \frac{1}{8} \sin 2t$$

is an antiderivative. So,

$$\frac{1}{2} \int_{-\pi/2}^{\pi/2} t \sin(2t) dt = \frac{t}{4} \cos 2t + \frac{1}{8} \sin 2t \Big|_{-\pi/2}^{\pi/2}$$

Which you can calculate.

Next, we find $\int_{-\pi/2}^{\pi/2} \frac{t^2}{2} (1 + \cos(2t)) dt = \int_{-\pi/2}^{\pi/2} \frac{t^2}{2} dt + \int_{-\pi/2}^{\pi/2} \frac{t^2}{2} \cos(2t) dt$.

The first integral is straightforward. To do $\int_{-\pi/2}^{\pi/2} \frac{t^2}{2} \cos(2t) dt$, use integration by parts twice.

Add this all up to find that the area enclosed by this curve is $\frac{\pi^3}{24}$.

Note that if you go through all of this for the incorrect orientation, but you still only trace out the curve once and the curve does not intersect itself except at the endpoints, your answer will be off by exactly a factor of -1 . So, if you get a negative answer, it is likely (although not guaranteed if you made other mistakes) that you simply had the wrong orientation, and the area is the absolute value of your answer.