MATH 231 U1, Spring 2009
Answers to HW 3, Section 6.3 problems 8, 10, 33, 34
Wednesday January 28th, 2009

6.3 #8: $\int \sin^4 x \; dx$

Here we must use the half angle formulas.

$$\begin{aligned}
\int \sin^4 x \; dx = \int (\sin^2 x)^2 \; dx = \int \frac{1}{4}(1-\cos 2x)^2 \; dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2(2x)) \; dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)) \; dx \\
&= \frac{1}{4}[x - \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x] + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C
\end{aligned}$$

#10: $\int \cot x \csc^4 x \; dx$

This can be done by substituting either $u = \csc x$ OR $u = \cot x$ (or by writing everything in terms of sines and cosines). Using the first substitution, we have $du = -\csc x \cot x \; dx$.

$$\int \cot x \csc^4 x \; dx = -\int u^3 \; du = -\frac{u^4}{4} + C = -\frac{\csc^4 x}{4} + C$$

If we use the second substitution we will get $\frac{-1}{2}\cot^2 x - \frac{1}{4}\cot^4 x + C$ (which is equal to our answer from the first method, can you prove why?)

#33: Show for any integer $n > 1$, we have the reduction formula

$$\int \sec^n x \; dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x \; dx$$

If $n = 2$, then $\int \sec^2 x \; dx = \tan x + C$, which is equal to the given formula for $n = 2$.
Let $n \geq 3$. Using integration by parts with $u = \sec^{n-2} x$ and $dv = \sec^2 x \; dx$ we have $du = (n-2)\sec^{n-3} x \sec x \tan x \; dx$, and $v = \tan x$ and we get

$$\int \sec^n x \; dx = \sec^{n-2} x \tan x - (n-2)\int \tan^2 x \sec^{n-2} x \; dx$$

Then, using $\tan^2 x = \sec^2 x - 1$ we get

$$\begin{aligned}
(n-2)\int \tan^2 x \sec^{n-2} x \; dx &= (n-2)\int (\sec^2 x - 1)\sec^{n-2} x \; dx \\
&= (n-2)\int sec^n x \; dx - (n-2)\int \sec^{n-2} x \; dx
\end{aligned}$$

Putting this together we get

$$\begin{aligned}\int \sec^n x \, dx &= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int sec^n x \, dx - (n-2) \int \sec^{n-2} x \, dx\end{aligned}$$

Now, add $(n-2) \int sec^n x \, dx$ to both sides and solve to get the answer.

#34: Evaluate (a) $\int \sec^3 x \, dx$ (b) $\int \sec^4 x \, dx$ (c) $\int \sec^5 x \, dx$.

(a) Using our formula we get: (see example 3.8 in section 6.3 for how to integrate $\sec x$).

$$\begin{aligned}\int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx \\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C\end{aligned}$$

(b) Using our formula we get:

$$\begin{aligned}\int \sec^4 x \, dx &= \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \int \sec^2 x \, dx \\ &= \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} \tan x + C\end{aligned}$$

(c) Using our formula and our answer to (a), we get:

$$\begin{aligned}\int \sec^5 x \, dx &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x \, dx \\ &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \Big(\frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x|\Big) + C\end{aligned}$$