MATH 231 U1, Spring 2009
Answers to HW 5, Section 6.3 problems 17, 20, 24, 36
Due Monday February 2nd, 2009

6.3 #17: $\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$

Just as in class, let $x = 3\sin\theta$ for θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ (we choose this domain for θ because it makes $\cos\theta > 0$, and thus $\sqrt{\cos^2\theta} = \cos\theta$). Then $dx = 3\cos\theta\ d\theta$ and substitute

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{9-x^2}}\,dx &= \int \frac{3\cos\theta\ d\theta}{9\sin^2\theta\sqrt{9-9\sin^2\theta}} \\
&= \frac{1}{9}\int \frac{3\cos\theta\ d\theta}{\sin^2\theta 3\sqrt{\cos^2\theta}} \\
&= \frac{1}{9}\int \frac{1}{\sin^2\theta}\ d\theta \\
&= \frac{1}{9}\int \csc^2\theta\ d\theta \\
&= -\frac{1}{9}\cot\theta + C
\end{aligned}$$

Now we must put our answer in terms of x. Drawing a reference triangle for $\frac{x}{3} = \sin\theta$, we can see that $\cot\theta = \frac{\sqrt{9-x^2}}{x}$, so our answer is $-\frac{\sqrt{9-x^2}}{9x} + C$

#20: $\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

For this problem we may use either trigonometric substitution or regular substitution. For the latter way, let $u = 4 - x^2$, then $du = -2x\ dx$ and $u(0) = 4$, $u(1) = 3$, so we get

$$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int_4^3 \frac{1}{\sqrt{u}}\ du = -\frac{1}{2}\int_4^3 u^{-1/2}\ du = -\frac{1}{2}[2u^{1/2}]_4^3 = \sqrt{4} - \sqrt{3} = 2 - \sqrt{3}$$

#24: $\int \frac{x}{\sqrt{x^2-4}}\,dx$

For this problem we may use either trigonometric substitution or regular substitution. For the former method, let $x = 2\sec\theta$ for θ in $(0, \pi/2)$ (we choose this domain because it makes $\tan\theta > 0$). Then $dx = 2\sec\theta\tan\theta\ d\theta$ and,

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2-4}}\,dx &= \int \frac{4\sec^2\theta\tan\theta}{2\sqrt{\sec^2\theta - 1}}\ d\theta \\
&= 2\int \frac{\sec^2\theta\tan\theta}{\tan\theta}\ d\theta \\
&= 2\int \sec^2\theta\ d\theta = 2\tan\theta + C = \sqrt{x^2-4} + C
\end{aligned}$$

Again, we may do the very last step by drawing a reference triangle for $\frac{x}{2} = \sec\theta$ and using it to calculate $\tan\theta = \frac{\sqrt{x^2-4}}{2}$ in terms of x...

#36 $\dfrac{4b}{a}\displaystyle\int_0^a \sqrt{a^2-x^2}\;dx$

We will use trigonometric substitution. Let $x = a\sin\theta$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then $dx = a\cos\theta\; d\theta$.
We will also switch the limits of integration to values of θ instead of values of x.
If $x = a$, then $a\sin\theta = a$, giving $\sin\theta = 1$, and so $\theta = \frac{\pi}{2}$.
If $x = 0$, then $a\sin\theta = 0$, giving $\sin\theta = 0$ (since $a > 0$), and so $\theta = 0$. This gives us:

$$\begin{aligned}
\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\;dx &= \frac{4b}{a}\int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\;d\theta \\
&= \frac{4b}{a}\int_0^{\pi/2} a\sqrt{1-\sin^2\theta}\;a\cos\theta\;d\theta \\
&= 4ba\int_0^{\pi/2} \sqrt{\cos^2\theta}\cos\theta\;d\theta \\
&= 4ba\int_0^{\pi/2} \cos^2\theta\;d\theta \\
&= \text{ ...use the half angle formula and antidifferentiate...} \\
&= 2ab\Big[\theta + \frac{\sin 2\theta}{2}\Big]_0^{\pi/2} \\
&= 2ab\Big[\frac{\pi}{2} + \frac{\sin\pi}{2} - (0+0)\Big] = ab\pi
\end{aligned}$$