MATH 231 U1, Spring 2009
Answers to HW 6, Section 6.4 problems 22, 23, 32
Due Wednesday February 4th, 2009

6.4 #22: Find the partial fraction decomposition and integrate $\frac{2x}{x^2-6x+9}$.

$$\frac{2x}{x^2-6x+9} = \frac{2x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

(This is how we start off for the case where we have a "repeated linear factor," in this case, (x-3) is our repeated linear factor.)
So, multiplying both sides by $(x-2)^2$ we get

$$2x = A(x-3) + B = Ax - 3A + B$$

Matching coefficients, we get the system

$$2 = A; \quad 0 = B - 3A$$

So, $A = 2$ and $B = 6$ and we get

$$\frac{2x}{x^2-6x+9} = \frac{2}{x-3} + \frac{6}{(x-3)^2}$$

You can check this by finding a common denominator and recombining these two fractions.
Now we integrate (you can do substitution with $u = x-3$ or just do them in your head).

$$\int \frac{2x}{x^2-6x+9}\,dx = \int \frac{2}{x-3}\,dx + \int \frac{6}{(x-3)^2}\,dx$$
$$= 2\ln|x-3| - \frac{6}{x-3} + C$$

#23: Find the partial fraction decomposition and integrate $\frac{x^3-4}{x^3+2x^2+2x}$.

Since the degrees of the polynomials from the numerator and denominator are the same, we start by using polynomial long division (or "synthetic division") (Come to office hours or review your notes if you don't know how to do this!) We get that

$$\frac{x^3-4}{x^3+2x^2+2x} = 1 - \frac{2x^2+2x+4}{x^3+2x^2+2x}$$

Now, we need to find the partial fraction decomposition of

$$\frac{2x^2+2x+4}{x^3+2x^2+2x} = \frac{2x^2+2x+4}{x(x^2+2x+2)}.$$

The quadratic polynomial x^2+2x+2 is irreducible (aka has no real roots, aka can't be factored over the real numbers. Do you know why this polynomial is irreducible? Hint: you may want to review the quadratic formula.) Since we have a both a linear factor and a quadratic factor, we need to write:

$$\frac{2x^2+2x+4}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$

Multiplying both sides by the denominator $x(x^2+2x+2)$ we get

$$2x^2+2x+4 = A(x^2+2x+2)+(Bx+C)x = Ax^2+2Ax+2A+Bx^2+Cx$$

Matching coefficients, we get the system

$$A+B=2; \quad 2A+C=2; \quad 4=2A$$

Solving, we get $A=2, B=0, C=-2$ thus,

$$\frac{x^3-4}{x^3+2x^2+2x} = 1 - \frac{2x^2+2x+4}{x^3+2x^2+2x} = 1 - \left(\frac{A}{x} + \frac{Bx+C}{x^2+2x+2}\right) = 1 - \frac{2}{x} - \frac{-2}{x^2+2x+2}$$

Now we integrate:

$$\begin{aligned}\int \frac{x^3-4}{x^3+2x^2+2x}\,dx &= \int 1\,dx - \int \frac{2}{x}\,dx + \int \frac{2}{x^2+2x+2}\,dx \\ &= x - 2\ln|x| + \int \frac{2}{x^2+2x+2}\,dx \\ &= x - 2\ln|x| + \int \frac{2}{(x+1)^2+1}\,dx \\ &= x - 2\ln|x| + 2\arctan(x+1) + C\end{aligned}$$

To do the last integral we completed the square in the denominator, and then used substitution with $u = x+1$ (or you could do it in your head if you've had enough practice.)

#32: Find the partial fraction decomposition for $\dfrac{x^3+2}{(x^2+1)^2}$ (but don't integrate). As suggested by problem 31, we write

$$\frac{x^3+2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}.$$

Multiplying both sides by $(x^2+1)^2$ we get

$$x^3+2 = (Ax+B)(x^2+1)+(Cx+D) = Ax^3+Ax+Bx^2+B+Cx+D$$

Matching coefficients gives the system
$A=1$ (from coeffs of x^3);
$B=0$ (from coeffs of x^2);
$A+C=0$ (from coeffs of x);
$B+D=2$ (from the constant terms)
Thus, $A=1$, $B=C=-1$ and $D=2$.
So, we have shown

$$\frac{x^3+2}{(x^2+1)^2} = \frac{x}{x^2+1} + \frac{-x+2}{(x^2+1)^2}$$

Remember, you can always check your answers!