

MATH 231 U1, Spring 2009  
Answers to HW 7, Worksheet 1, 2, 3  
Due Friday February 4th, 2009

#1. Compute  $\int_{-8}^{-4} \frac{x^3}{\sqrt{x^2 - 16}} dx$

We cannot compute this integral, because the integrand is not continuous on the interval  $[-8, -4]$ . This is the end of this problem.

But, to get some practice, lets change the limits of integration so that we can do the integral.

Try:  $\int_{-8}^{-4\sqrt{2}} \frac{x^3}{\sqrt{x^2 - 16}} dx.$

Notice that our integrand IS continuous on  $[-8, -4\sqrt{2}]$ .

Make the trigonometric substitution  $x = 4\sec\theta$ . Since we are looking at  $x$ -values which are all negative, we must choose  $\theta$ -values which give us negative  $x$ 's. Therefore, we must choose  $\theta$  in  $(\frac{\pi}{2}, \pi)$ .

Here, we will do the integral leaving off the limits of integration, then we will put our antiderivative in terms of  $x$  and then we will plug in our  $x$ -values. (The other option is to change your limits of integration to  $\theta$ -values now, and then you would not need to put the antiderivative in terms of  $x$  before you plug in.) Calculate  $dx = 4\sec\theta \tan\theta d\theta$  and substitute:

$$\int \frac{x^3}{\sqrt{x^2 - 16}} dx = \int \frac{4^3 \sec^3 \theta}{\sqrt{16 \sec^2 \theta - 16}} 4 \sec \theta \tan \theta d\theta \quad (1)$$

$$= 4^3 \int \frac{\sec^4 \theta \tan \theta}{\sqrt{\tan^2 \theta}} d\theta \quad (2)$$

$$= 4^3 \int \frac{\sec^4 \theta \tan \theta}{-\tan \theta} d\theta \quad (3)$$

$$= -4^3 \int \sec^4 \theta d\theta \quad (4)$$

$$= -4^3 \int \sec^2 \theta \sec^2 \theta d\theta \quad (5)$$

$$= -4^3 \int (\tan^2 \theta + 1) \sec^2 \theta d\theta \quad (6)$$

$$= -4^3 \int u^2 + 1 du \quad (7)$$

$$= -4^3 \left( \frac{u^3}{3} + u \right) + C \quad (8)$$

$$= -4^3 \left( \frac{\tan^3 \theta}{3} + \tan \theta \right) + C \quad (9)$$

From line (2) to line (3), notice that we replaced  $\sqrt{\tan^2 \theta}$  by  $-\tan \theta$ . This is because, for  $\theta \in (\frac{\pi}{2}, \pi)$ ,

$\tan \theta < 0$ , which means that  $\sqrt{\tan^2 \theta} = -\tan \theta$ .

To get from line (6) to line (7) we did a u-substitution with  $u = \tan \theta$

Now we are ready to put this antiderivative back in terms of  $x$ . We know  $\frac{x}{2} = \sec \theta$  for  $\theta \in (\frac{\pi}{2}, \pi)$ , and if we draw an appropriate reference triangle or do some algebra we can see that  $\tan \theta = -\frac{\sqrt{x^2-16}}{4}$ .

Here is the algebraic method:  $\frac{x}{4} = \sec \theta$  and  $-\tan \theta = \sqrt{\tan^2 \theta} = \sqrt{\sec^2 \theta - 1}$ . So,

$$-\tan \theta = \sqrt{\tan^2 \theta} = \sqrt{\frac{x^2}{16} - 1} = \frac{\sqrt{x^2 - 16}}{4}.$$

Therefore  $\tan \theta = -\frac{\sqrt{x^2-16}}{4}$ .

Thus we get that

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2-16}} dx &= -4^3 \left( \frac{-(x^2-16)^{3/2}}{3} - \frac{\sqrt{x^2-16}}{4} \right) \\ &= \left( \frac{(x^2-16)^{3/2}}{3} + 4^3 \frac{\sqrt{x^2-16}}{4} \right) \end{aligned}$$

#2. Write out a partial fraction decomposition for

$$\frac{x^5 - 4x^3 + 2}{(x+8)(x^3+x)^2(x-2)^4}$$

with the unknowns A, B, C etc. DO NOT solve for the A, B, C etc.

First, we must factor the bottom into distinct linear and irreducible quadratic factors. If a factor appears multiple times, we group those together. (For example, we write  $x^2$  instead of  $xx$ , but we must remember that this is a REPEATED LINEAR FACTOR, not an irreducible quadratic.)

$$\frac{x^5 - 4x^3 + 2}{(x+8)(x^3+x)^2(x-2)^4} = \frac{x^5 - 4x^3 + 2}{(x+8)x^2(x^2+1)^2(x-2)^4}$$

$$\frac{A}{x+8} + \frac{B}{x} + \frac{C}{x^2} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} + \frac{H}{x-2} + \frac{I}{(x-2)^2} + \frac{J}{(x-2)^3} + \frac{K}{(x-2)^4}$$

#3. a) Find the partial fraction decomposition for  $\frac{2x}{(x-1)(x^2+1)}$  (DO solve for the unknowns).

b) Calculate  $\int \frac{2x}{(x-1)(x^2+1)} dx$ .

a)

$$\begin{aligned} \frac{2x}{(x-1)(x^2+1)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \\ 2x &= Ax^2 + A + Bx^2 - Bx + Cx - C \end{aligned}$$

So,  $A + B = 0$ , so  $A = -B$ ,  $A - C = 0$ , so  $A = C$  and  $C - B = 2$ , so  $A + A = 2$ . Thus  $A = C = 1$  and  $B = -1$ .

$$\frac{2x}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{-x+1}{x^2+1}$$

b)

$$\begin{aligned} \int \frac{2x}{(x-1)(x^2+1)} dx &= \int \frac{1}{x-1} dx + \int \frac{-x+1}{x^2+1} dx \\ &= \ln|x-1| + \int \frac{1-x}{x^2+1} dx \\ &= \ln|x-1| + \int \frac{1}{x^2+1} dx - \int \frac{x}{x^2+1} dx \\ &= \ln|x-1| + \arctan x - \int \frac{x}{x^2+1} dx \\ &= \ln|x-1| + \arctan x - \frac{1}{2} \ln(x^2+1) + C \end{aligned}$$

#4. These were not part of the assignment to turn in, but I wanted to point out a helpful method for solving A, B, C etc. when you have linear factors (This was "writing exercise" 2 from 6.4).

Given, say,  $\frac{x^2+3x+1}{(x+2)(x-3)x} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{x}$ , notice that the roots of the denominator on the right side are  $-2, 3$  and  $0$

Multiply both sides by the denominator to get:

$$x^2 + 3x + 1 = A(x-3)x + B(x+2)x + C(x+2)(x-3)$$

Now notice that if we plug  $x = -2$  into the equation above, the two terms with an  $(x+2)$  in them become 0, giving

$$\begin{aligned} 4 - 6 + 1 &= A(-5)(-2) + 0 + 0 \\ -1 &= 7A \\ A &= -\frac{1}{7} \end{aligned}$$

Now if we plug in  $x = 0$  instead we get

$$\begin{aligned} 0 + 0 + 1 &= 0 + 0 + C(2)(-3) \\ 1 &= -6C \\ C &= -\frac{1}{6} \end{aligned}$$

And if we plug in the last root 3, we get

$$9 + 9 + 1 = 0 + B(5)(3) + 0$$

$$19 = 15B$$

$$B = \frac{19}{15}$$

This gives us an alternative method for finding  $A, B, C$  etc in some cases.