MATH 231 U1, Spring 2009
Answers to HW 8, Section 6.6, problems 8, 12, 15, 35
Due Friday February 4th, 2009

8. $\displaystyle\int_0^1 x^{-4/3}\,dx$

This integral is improper because $x^{-4/3} = \frac{1}{x^{4/3}}$ has a vertical asymptote at $x = 0$.

$$\begin{aligned}
\int_0^1 x^{-4/3}\,dx &= \lim_{R\to 0^+} \int_R^1 x^{-4/3}\,dx \\
&= \lim_{R\to 0^+} \Big[-3x^{-1/3} \Big]_R^1 \\
&= \lim_{R\to 0^+} [-3 + 3R^{-1/3}] \\
&= -3 + \lim_{R\to 0^+} \frac{3}{R^{1/3}} = DNE
\end{aligned}$$

Does not exist, because $\lim_{R\to 0+} \frac{3}{R^{1/3}}$ does not exist.

#12. $\displaystyle\int_1^5 \frac{2}{\sqrt{5-x}}\,dx$

This integral is improper because $\frac{2}{\sqrt{5-x}}$ has an asymptote at $x = 5$.

$$\begin{aligned}
\int_1^5 \frac{2}{\sqrt{5-x}}\,dx &= \lim_{R\to 5^-} \int_1^R \frac{2}{5-x}\,dx \\
&= \lim_{R\to 5^-} \Big[-4(5-x)^{1/2} \Big]_1^R \\
&= -4 \lim_{R\to 5^-} [(5-R)^{1/2} - 4^{1/2}] \\
&= -4 \lim_{R\to 5^-} [\sqrt{5-R} - 2] \\
&= -4(0-2) = 8
\end{aligned}$$

#15. $\displaystyle\int_0^3 \frac{2}{x^2-1}\,dx$

The asymptote here happens in the middle, at $x = 1$. So we must split this up in to two integrals.

$$\int_0^3 \frac{2}{x^2-1}\,dx = \int_0^1 \frac{2}{x^2-1}\,dx + \int_1^3 \frac{2}{x^2-1}\,dx$$

BOTH of the integrals on the right hand side of this equation are improper. We will start by investigating $\int_0^1 \frac{2}{x^2-1}\,dx$.

One way to do this integral is to find the partial fractions decomposition for $\frac{2}{x^2-1}$:

$$\frac{2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{-1}{x+1} + \frac{1}{x-1}$$

(I have omitted the steps for finding A and B.)

$$\begin{aligned}
\int_0^1 \frac{2}{x^2-1}\ dx &= \lim_{R\to 1^-} \int_0^R \frac{2}{x^2-1}\ dx \\
&= \lim_{R\to 1^-} \int (\frac{-1}{x+1} + \frac{1}{x-1})\ dx \\
&= \lim_{R\to 1^-} \Big[-\ln|x+1| + \ln|x-1| \Big]_0^R \\
&= \lim_{R\to 1^-} \Big[\ln|\frac{x-1}{x+1}| \Big]_0^R \\
&= \lim_{R\to 1^-} [\ln|\frac{R-1}{R+1}| - \ln|\frac{-1}{1}|] \\
&= \lim_{R\to 1^-} [\ln|\frac{R-1}{R+1}| - \ln 1] \\
&= \lim_{R\to 1^-} \ln|\frac{R-1}{R+1}| = DNE
\end{aligned}$$

This limit does not exist because as $R \to 1^-$, $|\frac{R-1}{R+1}|$ approaches 0,
so, as $R \to 1^-$, $\ln|\frac{R-1}{R+1}|$ approaches $-\infty$ (since $\lim\limits_{x\to 0} \ln x = -\infty$)

#35. Based on the answers to previous questions, conjecture a value r so that the integral $\int_0^a x^{-n}\ dx$ converges whenever $n < r$.

The answer is $r = 1$.

Lets see how we would prove this.

For $n \leq 0$ the integral is not improper, so it clearly converges (any proper integral converges.)

For $n = 1$

$$\begin{aligned}
\int_0^a x^{-1}\ dx &= \lim_{R\to 0^+} \int_R^a x^{-1}\ dx \\
&= \lim_{R\to 0^+} \Big[\ln x \Big]_R^a \\
&= \lim_{R\to 0^+} \ln a - \ln R \\
&= \ln a - \lim_{R\to 0^+} \ln R = -\infty = DNE
\end{aligned}$$

For $n > 1$ and $0 < n < 1$

$$\begin{aligned}\int_0^a x^{-n}\ dx &= \lim_{R\to 0^+} \int_R^a x^{-n}\ dx \\ &= \lim_{R\to 0^+} \left[\frac{x^{-n+1}}{-n+1}\right]_R^a \\ &= \lim_{R\to 0^+} [\frac{a^{-n+1}}{-n+1} - \frac{R^{-n+1}}{-n+1}] \\ &= \frac{a^{-n+1}}{-n+1} - \lim_{R\to 0^+} \frac{R^{-n+1}}{-n+1}\end{aligned}$$

so for $n > 1$

$$\int_0^a x^{-n}\ dx = \frac{a^{-n+1}}{-n+1} - \lim_{R\to 0^+} \frac{R^{-n+1}}{-n+1} = DNE$$

The limit does not exist in this case because $-n+1$ is a negative number, meaning $\lim_{R\to 0^+} R^{-n+1}$ does not exist (it is ∞).

and for $0 < n < 1$

$$\int_0^a x^{-n}\ dx = \frac{a^{-n+1}}{-n+1} - \lim_{R\to 0^+} \frac{R^{-n+1}}{-n+1}$$

The limit exists in this case because $-n+1$ is a positive number, meaning $\lim_{R\to 0^+} R^{-n+1}$ exists.