

MATH 231 U1, Spring 2009  
Answers to HW 9, Section 6.6, problems 44, 53-56  
Due Wednesday February 11th, 2009

#44. Does  $\int_1^{\infty} e^{-x^3} dx$  converge or diverge?

To answer this question, we will use the comparison test, and compare this integral to  $\int_1^{\infty} e^{-x} dx$ .

So, we must figure out whether  $\int_1^{\infty} e^{-x} dx$  converges. (This is a good one to remember!)

$$\begin{aligned}\int_1^{\infty} e^{-x} dx &= \lim_{R \rightarrow \infty} \int_1^R e^{-x} dx \\ &= \lim_{R \rightarrow \infty} \left[ -e^{-x} \right]_1^R \\ &= \lim_{R \rightarrow \infty} \left[ -e^{-R} + e^{-1} \right] \\ &= \lim_{R \rightarrow \infty} -e^{-R} + \frac{1}{e} = 0 + \frac{1}{e}\end{aligned}$$

So, we know that  $\int_1^{\infty} e^{-x} dx$  converges.

Now for the "comparison" part: we know that  $e^{x^3} \geq e^x$  for  $x$  in  $[1, \infty)$ . Therefore  $\frac{1}{e^{x^3}} \leq \frac{1}{e^x}$ , (which is the same as  $e^{-x^3} \leq e^{-x}$ )

Thus, by the comparison test,  $\int_1^{\infty} e^{-x^3} dx$  converges.

Questions 53-56 are True or False questions.

#53. If  $\lim_{x \rightarrow \infty} f(x) = 1$ , then  $\int_0^{\infty} f(x) dx$  diverges.

This is TRUE. To see an example of the intuition behind this, draw the function  $f(x) = 1$ . For this function,  $\lim_{x \rightarrow \infty} f(x) = 1$ , and it is plain to see that the area between  $f(x)$  and the  $x$ -axis on the interval  $[0, \infty)$  cannot possibly be finite. Below is a very useful fact which is related to this question.

FACT: If  $\int_0^{\infty} f(x) dx$  converges  $\lim_{x \rightarrow \infty} f(x) = 0$ .

#54. If  $\lim_{x \rightarrow \infty} f(x) = 0$ , then  $\int_0^{\infty} f(x) dx$  converges.

This is FALSE. The function  $f(x) = \frac{1}{x}$  is a counterexample. Can you think of other counterexamples?

Note: It is very important that you can tell the difference between this statement and the “FACT” I stated above.

#55. If  $\lim_{x \rightarrow 0} f(x) = \infty$ , then  $\int_0^1 f(x) dx$  diverges.

This is FALSE. The function  $f(x) = \frac{1}{\sqrt{x}}$  is a counterexample. Can you think of other counterexamples?

#56. If  $f(-x) = -f(x)$ , then  $\int_{-\infty}^{\infty} f(x) dx = 0$ .

This is FALSE. We must take a careful look at how  $\int_{-\infty}^{\infty} f(x) dx$  is defined.

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$$

Also, if either of the integrals on the right diverges, then the whole thing diverges.

Using this correct definition, it is easy to show that the odd function  $x^3$  is a counterexample, because clearly  $\int_{-\infty}^0 x^3 dx$  and  $\int_0^{\infty} x^3 dx$  are both divergent. (Really we only need one of them to be divergent to show the whole thing diverges.)