MATH 231 U1, Spring 2009
Exam 2: 8.1-8.5, Version 1
Wednesday March 18th, 2009

NAME: KEY
(Please Print)

Seat Assignment:

Back of Classroom

	1	2	3	4	5	6	7	
	1	2	3	4	5	6	7	
Windows	1	2	3	4	5	6	7	Door
	1	2	3	4	5	6	7	
	1	2	3	4	5	6	7	

Front of Classroom

DIRECTIONS:

- Do not open Exam until instructed to do so.
- Sit in the seat indicated above.
- Do each of the problems and show <u>all</u> work as the work leading to the solution is far more important than the actual answer.
- **NO WORK MEANS NO POINTS!**
- Calculators **ARE NOT ALLOWED** on this Exam.
- Box or circle and **LABEL** your final solution.
- You will have 50 minutes to complete this Exam.

1. (a) (6 points) Find the limit of the sequence $a_n = \dfrac{\ln(n^2)}{n}$.

$$\lim_{x\to\infty} \frac{\ln(x^2)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{2x}{x^2}\right)}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

has form $\frac{\infty}{\infty}$

So $\lim_{n\to\infty} \frac{\ln(n^2)}{n}$ must $= \boxed{0}$ too.

(b) (6 points) Show that the sequence $a_n = \dfrac{n!}{5^n}$ is monotonic for $n \geq 5$.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} = \frac{n+1}{5}$$ which is ≥ 1 for $n \geq 5$.

So $a_{n+1} \geq a_n$ and therefore the sequence is increasing. Thus it is monotonic.

2. For the following series, either *find the sum* of the series or *say why the series diverges.* (You DONT have to simplify. 6 points apiece)

(a) $\sum_{k=0}^{\infty} \frac{3^{k+1}}{8^k}$ $= 3 + \frac{3^2}{8} + \frac{3^3}{8^2} + \ldots$

is a geometric series with ratio $r = \frac{3}{8}$ and first term $a = 3$. $|r| = \frac{3}{8} < 1$ so this converges to $\frac{a}{1-r} = \boxed{\frac{3}{1-\frac{3}{8}}}$

(b) $\sum_{k=1}^{\infty} \frac{1}{5+2^{-k}}$

$$\lim_{k \to \infty} \frac{1}{5+2^{-k}} = \lim_{k \to \infty} \frac{1}{5+\frac{1}{2^k}} = \frac{1}{5} \neq 0$$

So this series $\boxed{\text{diverges}}$ by the k^{th} Term Test.

(c) $\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$ TELESCOPING!

$$S_n = \left(1 - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right) + \ldots + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right)$$

$$S_n = 1 - \frac{1}{\sqrt{n+1}}$$

So $$\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = \lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 - \frac{1}{\sqrt{n+1}} = \boxed{1}$$

3. (6 points) (a) State the Alternating Series Test. I have started it for you below:

Given an alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ where $a_k > 0$...

If $\lim_{k\to\infty} a_k = 0$ and $0 < a_{k+1} \leq a_k$

then the alternating series converges.

(b) (6 points) It is a fact that the series $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$ satisfies the hypotheses of the Alternating Series Test and converges to $S = \ln 2$. How close is $\sum_{k=1}^{99}(-1)^k \frac{1}{k}$ to $\ln 2$? That is, what is an upper bound on the error $|S - S_{99}|$ when approximating $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$ by S_{99}?

Your answer should be a real number.

$|S - S_n| \leq a_{n+1}$ by AST error estimate,

so $|S - S_{99}| \leq \frac{1}{100}$

$\therefore \; |\ln 2 - S_{99}| \leq \frac{1}{100}$

4. For the following series, *say whether the series converges or diverges* and why. (DO NOT try to calculate the sums of the series. Worth 5 points apiece.)

Comparison or LCT or Integral Test (a) $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

$$0 < \frac{1}{K^2+1} \le \frac{1}{K^2}$$ since $K^2+1 \ge K$ for $K \ge 1$.

and $\sum_{K=1}^{\infty} \frac{1}{K^2}$ is a <u>convergent</u> p-series ($p=2>1$)

So $\sum_{K=1}^{\infty} \frac{1}{K^2+1}$ $\boxed{\text{converges}}$ by the Comparison Test

LCT (b) $\sum_{k=1}^{\infty} \frac{k^5+5k^2-3}{2k^6+9k^3+2}$

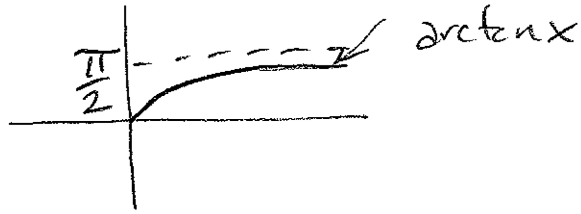
Let $b_K = \frac{1}{K}$, we know $\sum_{K=1}^{\infty} \frac{1}{K}$ is <u>divergent</u>.

Use the LCT:

$$\lim_{K\to\infty} \frac{a_K}{b_K} = \lim_{K\to\infty} \frac{K^5+5K^2-3}{2K^6+9K^3+2} \cdot \frac{K}{1} = \lim_{K\to\infty} \frac{K^6+5K^3-3K\left(\frac{1}{K^6}\right)}{2K^6+9K^3+2\left(\frac{1}{K^6}\right)}$$

$$\lim_{K\to\infty} \frac{1+\frac{5}{K^3}-\frac{3}{K^5}}{2+\frac{9}{K^3}+\frac{2}{K^6}} = \frac{1}{2} \boxed{>0}$$

So, by the LCT, since $\sum_{K=1} \frac{1}{K}$ diverges, this series diverges too.



Problem 4 continued: (For the following series, *say whether the series converges or diverges* and why.)

(c) $\sum_{k=1}^{\infty} \frac{\arctan(k)}{1+k^2}$

Integral Test OR LCT or comparison

OR

Comparison Test:

$0 \le \arctan(k) < \frac{\pi}{2}$ for $k \ge 1$

So $0 \le \frac{\arctan(k)}{1+k^2} < \frac{\frac{\pi}{2}}{1+k^2} \le \frac{\frac{\pi}{2}}{k^2}$

$\sum_{k=1}^{\infty} \frac{\pi/2}{k^2} = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$ which conv. by p-test

$\therefore \sum^{\infty} \frac{\arctan(k)}{1+k^2}$ conv. by comparison Test

Integral Test: $\frac{\arctan(x)}{1+x^2}$ is a continuous function, it is ≥ 0 and decreasing on $[1,\infty)$ (since $0 \le \arctan(x) < \frac{\pi}{2}$ and $1+x^2$ increases on $[1,\infty)$)

$$\int_1^{\infty} \frac{\arctan x}{1+x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{\arctan x}{1+x^2}\,dx$$

$u = \arctan x$
$du = \frac{1}{1+x^2}\,dx$

$$= \lim_{R\to\infty} \int_{\arctan 1}^{\arctan R} u\,du$$

$$= \lim_{R\to\infty} \frac{1}{2} u^2 \Big|_{\arctan 1}^{\arctan R}$$

$$= \lim_{R\to\infty} \frac{1}{2}(\arctan R)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

$$= \lim_{R\to\infty} \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2$$

So the $\int$ converges, $\therefore$ the series converges

(d) $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ Look at abs. values:

$$\sum_{k=1}^{\infty} \frac{|\cos k|}{k^3}$$

$0 \le |\cos k| \le 1$

so $0 \le \frac{|\cos k|}{k^3} \le \frac{1}{k^3}$

and $\sum_{k=1}^{\infty} \frac{1}{k^3}$ conv. by p-test. So $\sum_{k=1}^{\infty} \frac{|\cos k|}{k^3}$ converges, and $\therefore$ $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ converges (absolutely)

5. (6 points apiece)

(a) What does does it mean for a series $\sum_{k=1}^{\infty} a_k$ to *convergence conditionally*? (Give the definition.)

$\sum_{k=1}^{\infty} a_k$ <u>converges</u> and $\sum_{k=1}^{\infty} |a_k|$ diverges

(a.k.a. $\sum^{\infty} a_k$ DOES NOT converge absolutely)

Problem 5 continued:
Tell me whether the following series are *absolutely convergent, conditionally convergent or divergent* and why. Be sure to cite any tests you are using.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

$\lim_{k \to \infty} \frac{1}{\sqrt{k}} = 0$ and $0 < \frac{1}{\sqrt{k+1}} \leq \frac{1}{\sqrt{k}}$ for $k \geq 1$

since $\sqrt{k+1} \geq \sqrt{k} > 0$.

So this series converges by the AST.

Does $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{\sqrt{k}} \right|$ converge? it $= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ which

diverges by the p-test $(p = \frac{1}{2} < 1)$

So this $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges conditionally

(c) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{4^k}{(2k)!}$

RATIO TEST:

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{(-1)^{k+2} 4^{k+1}}{(2(k+1))!} \cdot \frac{(2k)!}{(-1)^{k+1} 4^k} \right| = \lim_{k \to \infty} \frac{4 \cancel{(2k)!}}{(2k+2)(2k+1)\cancel{(2k)!}}$$

$$= \lim_{k \to \infty} \frac{4}{(2k+2)(2k+1)} = 0 < 1$$

So the series is abs. conv. by the Ratio Test.

6. For these conceptual questions, be sure to (briefly) explain your answers. You will not receive credit without some explanation.
For instance, if you think the statement is true, be sure to cite a test or theorem which supports that answer. If you think the statement is false, give a counterexample, draw a picture or cite a test or theorem. (a)-(c) are True or False, (d) is a short answer question. (20 points total)

(a) True or False: If $\lim_{k\to\infty} a_k = 0$ then the series $\sum_{k=1}^{\infty} a_k$ converges.

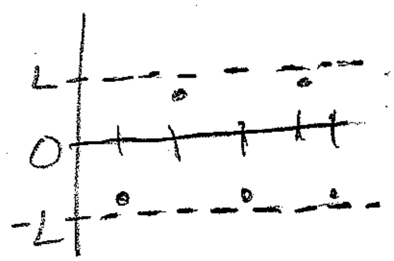
FALSE. $\sum_{k=1}^{\infty} \frac{1}{k}$ is a counterexample.

(b) True or False: For any number L, if $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} (-1)^n a_n = L$.

FALSE

If $L \neq 0$, we get a counterexample

e.g.



The odd terms have limit $-L$

the even terms have limit L

and for $L \neq 0$

$L \neq L$

Problem 6 continued:

(c) True or $\boxed{\text{False:}}$ If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both diverge then $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges.

FALSE The series from question 2(c)

$$\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}} - \frac{1}{\sqrt{K+1}}$$

is a counterexample.

(d) Assume the series $\sum_{k=1}^{\infty} a_k$ is conditionally convergent.

(i) What is $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$?

$\lim_{K\to\infty} \left|\frac{a_{K+1}}{a_K}\right| = 1$ since if it is <1 then the series conv. abso by the Ratio test, and if it is >1 the series diverges by the Ratio Test.

(ii) What is $\lim_{k\to\infty} |a_k|$?

$\lim_{K\to\infty} |a_K| = 0$, since otherwise the series $\sum_{K=1}^{\infty} a_K$ would diverge by the K^{th} Term test

Extra Credit: Write the number $1.\overline{2009} = 1.2009200920092009...$ as a ratio of integers (a.k.a. a fraction)
You must show appropriate work here to get credit. Worth 5 points, all or nothing.

$$1.\overline{2009} = 1 + \frac{2009}{10^4} + \frac{2009}{10^8} + \frac{2009}{10^{12}} + \cdots$$

$$= 1 + \sum_{k=0}^{\infty} \frac{2009}{10^4} \cdot \left(\frac{1}{10^4}\right)^k \leftarrow \text{geom. series w/}$$

$r = \frac{1}{10^4}$ $|r| = \frac{1}{10^4} < 1$

$a = \frac{2009}{10^4}$

$$= 1 + \frac{\frac{2009}{10^4}}{1 - \frac{1}{10^4}}$$

$$= 1 + \frac{2009}{10^4 - 1} = \frac{10^4 - 1}{10^4 - 1} + \frac{2009}{10^4 - 1}$$

$$= \frac{(10^4 - 1) + 2009}{10^4 - 1} = \frac{9999 + 2009}{9999}$$

$$\boxed{= \frac{12008}{9999}}$$