MATH 231 U1, Spring ~~2009~~
Exam 3: 8.6-8.8, 9.1-9.~~3~~ Version 2
Friday April 24th, ~~2009~~

Watch out for slight differences for Version 1!

NAME: ANSWERS
(Please Print)

Seat Assignment:

Back of Classroom

Windows | Door

1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7
1	2	3	4	5	6	7

Front of Classroom

DIRECTIONS:

- Do not open Exam until instructed to do so.
- Sit in the seat indicated above.
- Do each of the problems and show <u>all</u> work as the work leading to the solution is far more important than the actual answer.
- **NO WORK MEANS NO POINTS!**
- Calculators **ARE NOT ALLOWED** on this Exam.
- Box or circle and **LABEL** your final solution.
- You will have 50 minutes to complete this Exam.

1. (15 points) (a) Find the **interval of convergence** for the power series $\sum_{k=0}^{\infty}(x+5)^k$. **To what rational function does this power series converge?**

geometric with ratio $x+5$

So, converges iff $|x+5|<1$ $\Rightarrow$ I.O.C. $=(-6,-4)$

$-1 < x+5 < 1$

It converges to $\frac{1}{1-(x+5)}$.

(b) Find the interval of convergence for the power series $\sum_{k=1}^{\infty}\frac{(x-6)^{2k}}{\sqrt{k}3^k}$.

Ratio test:

$$\lim_{k\to\infty}\left|\frac{(x-6)^{2k+2}}{\sqrt{k+1}\,3^{k+1}}\cdot\frac{\sqrt{k}\,3^k}{(x-6)^{2k}}\right| = \lim_{k\to\infty}\sqrt{\frac{k}{k+1}}\cdot\frac{1}{3}\cdot|x-6|^2$$

Which is <1 iff $|x-6|^2<3$

$|x-6| < \sqrt{3}$

$-\sqrt{3} < x-6 < \sqrt{3}$

Test endpts:

x in $(6-\sqrt{3}, 6+\sqrt{3})$

If $x=6\pm\sqrt{3}$ then

$$\sum_{k=1}^{\infty}\frac{(x-6)^{2k}}{\sqrt{k}\,3^k} = \sum_{k=1}^{\infty}\frac{(\pm\sqrt{3})^{2k}}{\sqrt{k}\,3^k} = \sum_{k=1}^{\infty}\frac{3^k}{\sqrt{k}\,3^k} = \sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$$

which diverges by the p-test. So the I.O.C. is $(6-\sqrt{3}, 6+\sqrt{3})$

2. (15 points) (a) State Taylor's Theorem.

Suppose $f(x)$ has $(n+1)$ derivatives ...

look it up in the book!

(b) Find $P_3(x) =$ the Taylor Polynomial of degree 3 for the function $f(x) = \sqrt[3]{x}$ centered at 1.

$f(x) = x^{1/3}$ $\quad f(1) = 1$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f'(1) = \frac{1}{3}$

$f''(x) = \frac{-2}{9}x^{-5/3}$ $\quad f''(1) = \frac{-2}{9}$

$f^{(3)}(x) = \frac{10}{27}x^{-8/3}$ $\quad f^{(3)}(1) = \frac{10}{27}$

$$\boxed{P_3(x) = 1 + \frac{1}{3}(x-1) - \frac{2}{9\cdot 2!}(x-1)^2 + \frac{10}{27\cdot 3!}(x-1)^3}$$

(c) Estimate the error you would have if you used $P_3(x)$ from part (b) to approximate $\sqrt[3]{0.99}$. (DO NOT actually calculate the approximation.) You may use the fact that $|f^{(4)}(z)| \leq 1.05$ for all $z \in (0.99, 1)$, and DONT simplify.

by Taylor's Thm

$$|R_3(.99)| = \frac{|f^{(4)}(z)|}{4!}|0.99-1|^4 \leq \frac{1.05}{4!}(0.01)^4$$

for some z in $(0.99, 1)$

3. (15 points) (a) Find the Taylor series for $\frac{1}{1+5x^2}$ centered at 0.

this looks like $\frac{a}{1-r}$ with $a=1$, $r=-5x^2$

so, it is the sum of the geom. series

$$\sum_{K=0}^{\infty}(-5x^2)^K = \boxed{\sum_{K=0}^{\infty}(-1)^K 5^K x^{2K}}$$

(b) Starting from the series you found in part (a), find the Taylor series for $f(x) = \frac{10x}{(1+5x^2)^2}$ centered at 0.

$$\frac{d}{dx}\left(\frac{1}{1+5x^2}\right) = \frac{-10x}{(1+5x^2)^2}$$

So $\frac{10x}{(1+5x^2)^2} = \frac{-d}{dx}\left(\sum_{K=0}^{\infty}(-1)^K 5^K x^{2K}\right) = -\sum_{k=0}^{\infty}(-1)^K 5^K 2K x^{2K-1}$

$$= \boxed{\sum_{K=0}^{\infty}(-1)^{K+1} 5^K 2K\, x^{2K-1}}$$

4. (10 points) Find the value of the limit

$$\lim_{x \to 0} \frac{\sin(x^2) - x^2 e^x}{3x^3}$$

by using Taylor series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots \qquad x^2 e^x = x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{3!} + \cdots$$

$$\lim_{x \to 0} \frac{\sin(x^2) - x^2 e^x}{3x^3} = \lim_{x \to 0} \frac{\left(\cancel{x^2} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right) - \left(\cancel{x^2} + x^3 + \frac{x^4}{2} + \cdots\right)}{3x^3}$$

$$= \frac{1}{3} \lim_{x \to 0} \left(\frac{x^3}{3!} + \frac{x^7}{5!} + \cdots\right) - \left(1 + \frac{x}{2} + \cdots\right) = \frac{1}{3}(-1) = \frac{-1}{3}$$

Since all the terms containing powers of x will go to 0 as x goes to 0.

5. (10 points) For the following True or False questions, be sure to BRIEFLY JUSTIFY your answer. (Cite a theorem, give an example or explain your reasoning briefly.)

(a) True or False: The Taylor series for $\ln x$ centered at 1 can be used to approximate $\ln 4$.

FALSE the I.O.C. of that series is $[0,2)$, which doesn't contain 4.

(b) True or False: Because the radius of convergence for the Maclaurin series for $\arctan(x)$ is 1, we know the radius of convergence for the Maclaurin series of $\arctan(-5x)$ is $\frac{1}{5}$.

TRUE

since $|-5x| < 1 \Longleftrightarrow |x| < \frac{1}{5}$

(we did a HW question like this)

(c) True or False: Given a power series $\sum_{k=0}^{\infty} b_k(x-c)^k$, if you do the ratio test and find that

$$\lim_{k\to\infty} \left| \frac{b_{k+1}(x-c)^{k+1}}{b_k(x-c)^k} \right| = 0$$

then the power series diverges for all x.

FALSE

In this case the R.O.C. is ∞, so the seric converges for all x

(d) True or False: the *fastest* path between two points is always a straight line.

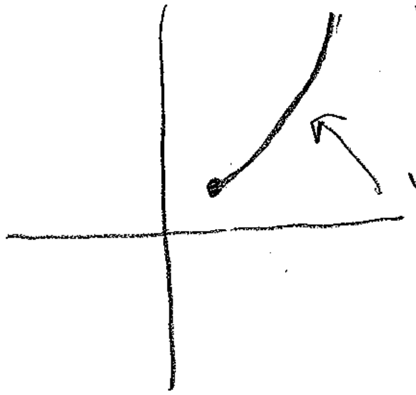
FALSE

Sometimes it is a cycloid.

6. (25 points)(a) Sketch the curve: $x(t) = t^2 + 1, \quad y(t) = t^4 + 2t^2 + 1.$

$$x - 1 = t^2 \qquad y = (x-1)^2 + 2(x-1) + 1 = x^2$$

$$\boxed{x \geq 1}$$



$y = x^2$ for $x \geq 1$

(b) Find the slope of the tangent line to the curve from part (a) at the point $(4, 16)$. (Do not simplify.)

$$x'(t) = 2t \qquad y'(t) = 4t^3 + 4t$$

$$\underline{\text{so}} \quad \left.\frac{dy}{dx}\right|_{t=c} = \frac{4t^3 + 4t}{2t}$$

when $x = 4$ and $y = 16$, $t = \sqrt{3}$ so,

$$\left.\frac{dy}{dx}\right|_{(4,16)} = \left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{4(\sqrt{3})^3 + 4\sqrt{3}}{2\sqrt{3}} = 8.$$

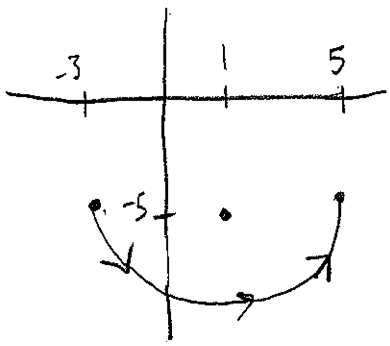
(c) Write down (but do not evaluate) an integral which gives you the surface area of the surface of revolution you get from revolving the segment of the curve from part (a) for $0 \leq t \leq 3$ about the line $y = -1$.

$$\int_0^3 2\pi \left|y(t) - (-1)\right| \sqrt{(2t)^2 + (4t^3 + 4t)^2}\, dt$$

$$= \int_0^3 2\pi (t^4 + 2t + 2) \sqrt{(2t)^2 + (4t^3 + 4t)^2}\, dt$$

7. (10 points) Find parametric equations for the BOTTOM HALF of the circle of radius 4 with center $(1, -5)$. Then, **sketch** this curve and indicate its **orientation** as given by your parametric equations.

For example:



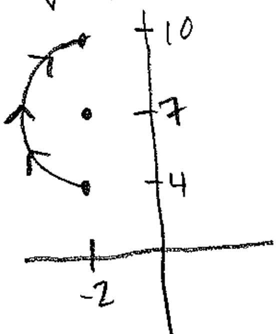
$x(t) = 4\cos t + 1$

$y(t) = 4\sin t - 5$

for t in $[\pi, 2\pi]$

Version 1

For example:



$x(t) = 3\cos t - 2$

$y(t) = 3\sin t - 7$

for t in $[-\frac{\pi}{2}, \frac{\pi}{2}]$

TEST ENDS HERE

Extra credit: (5 points all or nothing)
Suppose

$$f(x) = x - x^8 + 3x^{16} + \dots \quad \text{for} \quad |x| < 1,$$

and

$$g(x) = 2x - 2x^9 + x^{17} + \dots \quad \text{for} \quad |x| < 1.$$

Let $h(x) = f(x)g(x)$. Find $h^{(17)}(0)$. Justify your answer. Do not simplify.

The coefficient of x^{17} in $f(x) \cdot g(x)$ will be $2 + 6 = 8$

from $x^8 \cdot 2x^9$ from $2x \cdot 3x^{16}$

So $\frac{h^{(17)}(0)}{17!} = 8 \Rightarrow h^{(17)}(0) = 8 \cdot 17!$