MATH 231 U1, Spring 2009
Quiz 9, Version 1
Friday April 10th, 2009

1. (a) What is the *formula* for the Taylor series of $f(x)$ centered at c?

$$\sum_{K=0}^{\infty} \frac{f^{K}(C)}{K!}(X-C)^{K}$$

(b) State Taylor's Theorem.

Assuming f has at least $n+1$ derivatives on $(C-R, C+R)$ for some $R>0$, for X in $(C-R, C+R)$ the error $R_n := f(x) - P_n(x)$ when approx. $f(x)$ by $P_n(x)$

is: $R_n(X) = \dfrac{f^{(n+1)}(Z)}{(n+1)!}(x-c)^{n+1}$ for some Z between X and C.

2. Let $f(x)$ have Taylor series expansion $\sum_{k=0}^{\infty} b_k(x-1)^k$ centered at 1, and assume the I.O.C. for this series is $(-\infty, \infty)$.
Give an upper bound for the error $|R_3(2)|$ in using $P_3(x)$ to approximate $f(2)$.
Also assume that

$$|f^{(4)}(z)| \leq 6.$$

By Taylor's Theorem

because $|f^{4}(z)| \leq 6$ is given

$$|R_3(2)| = \left|\frac{f^{4}(Z)}{4!}(2-1)^4\right| \leq \frac{6}{4!}(2-1)^4 = \frac{1}{4}$$

for some Z in $(1,2)$

3. Use power series to find the following limit. (Hint: don't try to find the relevant Taylor series from scratch, instead, start from series you memorized.)

Since $x \to 0$, we will use series centered at 0.

$$\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2}$$

We know the series for $\cos x$ centered at 0 is $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ which has I.O.C. $(-\infty, \infty)$

The series for e^x centered at 0 is $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ (I.O.C. = $(-\infty,\infty)$)

so $\boxed{e^{x^2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}}$ for x in $(-\infty, \infty)$

(you also should show how to get this series)

THIS is the (approximate) amount of work to show:

$$\lim_{x\to 0} \frac{\cos x - e^{x^2}}{x^2} = \lim_{x\to 0} \frac{\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} - \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) - \left(1 + \frac{x^2}{1} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right)}{x^2}$$

$$= \lim_{x\to 0} \frac{\left(\frac{-1}{2} - 1\right)x^2 + \left(\frac{1}{4!} - \frac{1}{2}\right)x^4 + \cdots}{x^2}$$

$$= \lim_{x\to 0} -\frac{3}{2} + \left(\frac{1}{4!} - \frac{1}{2}\right)x^2 + \cdots$$

all these terms have the form (constant) x^{2k} them, so they $\to 0$ as $x \to 0$

$$= \frac{-3}{2}$$