MATH 231 U1, Spring 2009
What The Heck is Up With Sequences?
A Summary of Tools in Section 8.1

This worksheet is intended to help you organize the "tools" (definitions, theorems and concepts) we learned in Section 8.1. It is NOT completely comprehensive, but I hope it helps. Also, remember that there may be many ways to attack any given problem.

Directions: Go through your book and your notes in order to fill in the blanks below. For convenience, I have assumed all the sequences start with $n = 1$. I have included lots of examples and exercises to try.

1. What is a sequence? (Give the definition and some examples)

2. What is the definition of convergence for a sequence? (Definition 1.1)

3. What does it mean for a sequence to be increasing? decreasing?

4. What does it mean for a sequence to be monotonic?

How can you prove a sequence is monotonic? (See Examples, 1.10, 1.11, 1.13)

- If the sequence clearly jumps back and forth between begin positive and negative terms (alternates sign) then it CAN'T be monotonic, because it will be neither increasing nor decreasing.
- For a positive sequence $\{a_n\}_1^\infty$ (where $a_n > 0$ for all n), look at $\frac{a_{n+1}}{a_n}$. If this quantity is < 1 then $a_{n+1} < a_n$ and if it is > 1 then $a_{n+1} > a_n$.
- For a positive sequence $\{a_n\}_1^\infty$, you can also try to look at $a_{n+1} - a_n$. If this quantity is positive then $a_{n+1} > a_n$ and if it is negative then $a_{n+1} < a_n$.
- If the terms in your sequence are all negative after some n, you could look at the sequence $|a_n|$ to discover whether it is increasing after some n. If $|a_n|$ is increasing, then your original a_n is decreasing.
- Otherwise, be creative! Look at the general terms a_{n+1} and a_n and see if there is any reason one is always smaller than the other.

Example exercises: 33-38, 23 (show the seq. in #23 is decreasing for $n \geq 3$), 25

5. What does it mean for a sequence to be bounded?

How can you tell if a sequence is bounded? (See Example 1.12)

- One way is to use your reasoning to construct a chain of inequalities $\leq$ which begins with $|a_n|$ and ends with a real number M.
- If a_n is positive for every $n \geq 1$ and you already know that the sequence a_n is *decreasing*, then clearly the sequence is bounded by a_1. (See Example 1.13)

Example exercises: 39-42

6. What happens to the sequence $a_n = \frac{1}{n^p}$ for $p > 0$?

FACTS: In general, if you have a sequence of the form

$$\left\{\frac{a_n}{b_n}\right\}_1^\infty$$

and the limit of the numerator is $\lim_{n\to\infty} a_n = L$ and the limit of the denominator is $\lim_{n\to\infty} b_n = \infty$ (or $-\infty$), then you know that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

But, if the limit of the numerator is $\lim_{n\to\infty} a_n = \infty$ and the limit of the denominator is $\lim_{n\to\infty} b_n = L$ then

$$\lim_{n\to\infty} \frac{a_n}{b_n} \text{ does not exist}$$

The sequence will go to ∞ or $-\infty$.

You may use these facts when you are doing problems on homework, quizzes and exams.
Example exercises: 5, 6, 9, 10, 13, 14

7. Using algebra to rearrange or simplify the general term a_n is one major tool we use to figure out limits of sequences. See Examples 1.3, 1.4
Example exercises: 7, 8, 11-17, 21, 26

8. What are the main theorems for working with sequences?
(State Theorems 1.1, 1.2, 1.3 and 1.4, and Corollary 1.1)

9. Using Theorem 1.2 with what you already know from Calc I about limits of functions, and specifically L'Hopital's rule, can be a powerful tool for finding limits of sequences. See Example 1.6.
(Remember: not every sequence has an "associated function." For example, there is NO function $f(x)$ such that $f(n) = a_n = \frac{5^n}{n!}$. You will need to use other methods for sequences in cases like this.)
Example exercises: 7, 8, 11-14, 21-23

10. Squeeze Theorem (Example 1.7)

Example exercises: 29-32, 20, 24

11. Theorem 1.4: monotonic + bounded $\implies$ convergent (See Example 1.13)
Note that showing a sequence is bounded and monotonic tells you that it converges, but it does not tell you WHAT it converges to!

Example exercises: for 37, 38, 42 show the sequence converges by showing it is bounded and monotonic.

12. Sequences with alternating signs (See Example 1.5, 1.7)

- Look at $|a_n|$ and try to apply Corollary 1.1
- Try to apply the Squeeze Theorem
- **FACT:** If your sequence $\{a_n\}_1^\infty$ *alternates* between positive values and negative values and $\lim_{n\to\infty} |a_n| = L \neq 0$ or $\lim_{n\to\infty} |a_n|$ diverges, then $\{a_n\}_1^\infty$ diverges.

Example exercises: 15-17, 18, 24, 29-32

Tail end comment
Lastly, note that when we are trying to figure out $\lim_{n\to\infty} a_n$, we may ignore finitely many terms at the beginning, because what matters is what happens as $n \to \infty$. So, for example, if a sequence $\{a_n\}_1^\infty$ is increasing and bounded for $n \geq 100$, we still may use Theorem 1.4 and conclude that it converges.