Taylor's Theorem

Calc II: Taylor's Theorem

\[ f: \mathbb{R} \to \mathbb{R} \text{ is } C^\infty. \text{ For any } a \in \mathbb{R}, \]

\[ f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(a,x), \]

where \( R_k(a,x) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1} \text{ for some } z \in [a,x]. \)

Recall: \( T_k(x) \) has same derivatives as \( f(x) \) at \( x=a \) up to order \( k \).

Disc: Taylor polynomials for \( n \)-variable \( f(x) \).

If \( f: \mathbb{R}^n \to \mathbb{R} \text{ is } C^1 \), we already know \( T_1(x) \).

\[ T_1(x) = f(a) + \nabla f(a) \cdot (x-a) \]

the tangent hyperplane to surface \( z = f(x) \) at \( (a,f(a)) \) or \( L_{\nabla f(a)}(x) \).

Ex: \( f(x,y) = e^{-x^2-y^2} \)

a) 1st order approx about \( (0,0) \):

\[ f(0,0) = e^0 = 1 \]

\[ f_x(0,0) = -2xe^{-x^2-y^2} \]

\[ f_y(0,0) = -2ye^{-x^2-y^2} \]

\[ f_{xx}(0,0) = -2xe^{-x^2-y^2} \quad \text{and} \quad f_{yy}(0,0) = -2ye^{-x^2-y^2} \]

\[ f_{xy}(0,0) = -2e^{-x^2-y^2} \]

\[ T_1(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = e^{-x^2-y^2} \]

\[ \nabla f(0,0) = (-2x, -2y) \]

\[ L_{\nabla f(0,0)}(x,y) = -2x e^{-x^2-y^2} x - 2y e^{-x^2-y^2} y \]
\[ T_{1}(x,y) = L_{1,0,0}(x,y) \]
\[ = f_{1,0,0} + f_{x,0,0} (x-0) + f_{y,0,0} (y-0) \]
\[ = 1 + 0 + 0 \]
\[ = 1. \]

b) What about a 2nd-order approx?

Idea: require all 2nd-order partials to match.

Recall \( f_{xx}(x,y) = -de^{-x^2+y^2} + 4x^2e^{-x^2+y^2} \)
\( f_{xy}(x,y) = f_{yx}(x,y) = 4xy e^{-x^2+y^2} \)
\( f_{yy}(x,y) = -de^{-x^2+y^2} + 4y^2 e^{-x^2+y^2} \)

At \((0,0)\), \( f_{xx}(0,0) = -d \)
\( f_{xy}(0,0) = f_{yx}(0,0) = 0 \)
\( f_{yy}(0,0) = -d. \)

What polynomial term needs to be associated with each 2nd-order partial?
\( f_{xx} : \) For \( f_x(x,y) \), need two derivatives in \( x \).

Candidate: \( f_{x,0,0}(x-0)(x-0) = f_{x,0,0} x^2 \)

Is this right?

\[
\frac{d^2}{dx^2} (f_{x,0,0} x^2) = 2 f_{x,0,0}
\]

As with Calc I, need to account for scalars due to power rule.

Again, use \( f_{x,0,0} (x-0)^2 \).

\( f_y : \) Same discussion.

Use \( f_{y,0,0} (y-0)^2 \).

\( f_{xy} / f_{yx} : \)

For \( f_{xy} \), candidate \( f_{xy}(0,0) (x-0)(y-0) \).

This seems good:

\[
\frac{\partial}{\partial x} \left( \frac{1}{2} (f_{xy}(0,0) x y) \right) = f_{xy}(0,0)
\]

but here is a problem.
If we do the same for \( f_{yx} \),

\[ f_{yx}(0,0) = (y-o)(x-o) \text{. Then our candidate for } T_2 \text{ has the mixed terms} \]

\[ f_{xy}(0,0)x + f_{yx}(0,0)xy = 2f_{xx}(0,0)xy \]

by Clairaut's Thm.

So, again we require the scalar \( \frac{1}{2} \).

All together,

\[
\frac{1}{\sigma^2} x^2 y^2 \approx T_2(x,y)
\]

\[
= f(0,0) + f_x(0,0)(x-o) + f_y(0,0)(y-o) + \frac{1}{d^3} f_{xx}(0,0)(x-o)^2 + \frac{1}{d^3} f_{yy}(0,0)(y-o)^2 + \frac{1}{d^2} f_{xy}(0,0)(x-o)(y-o) + \frac{1}{d^2} f_{yx}(0,0)(x-o)(y-o) + \frac{1}{d^3} f_{xy}(0,0)(x-o)(y-o)
\]

\[
= 1 + 0 + 0 + \frac{1}{d} (-d^2) + 0 + 0 + \frac{1}{d} (-d^2)
\]

\[
= 1 - x^2 - y^2 .
\]
Thm: $2^{nd}$-order Taylor Thm

Let $f: \mathbb{R}^n \to \mathbb{R}$ be $C^2$ at $\vec{x} = \vec{a}$, then define

$$T_2(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \frac{1}{2} \sum_{i,j=1}^{n} f_{ij}(\vec{a}) (\vec{x} - \vec{a})^i (\vec{x} - \vec{a})^j$$

Then $f(\vec{x}) = T_2(\vec{x}) + R_2(\vec{x}, \vec{a})$ where $\frac{R_2(\vec{x}, \vec{a})}{||\vec{x} - \vec{a}||^2} \to 0$ as $\vec{x} \to \vec{a}$.

Ex: Compute the $2^{nd}$-order Taylor formula for $f(x,y) = e^{x+y} \cos xy$ about $(0,0)$.

Soln: $f(0,0) = 1$

$\nabla f(x,y) = (e^x \cos y, -e^x \sin y)$

$\nabla f(0,0) = (1, 0)$

$f_{xx} = e^x \cos y$, $f_{xx}(0,0) = 1$

$f_{xy}(x,y) = -e^x \sin y$, $f_{xy}(0,0) = 0$

Since $f$ is $C^2$, $f_{yx}(0,0) = 0$ also.

$f_{yx}(x,y) = -e^x \cos y$, $f_{yx}(0,0) = -1$
\[
T_j(x,y) = f(0,0) + Df(0,0) \cdot (x-0, y-0) \\
+ \frac{1}{2} \left[ f_{xx}(0,0) (x-0)^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(0,0) (y-0)^2 \right] \\
= 1 + \left( \frac{1}{10} \cdot (x,y) \right) + \frac{1}{2} \left[ \frac{1}{10} x^2 + \frac{1}{10} x y + (-1) y^2 \right] \\
= 1 + x + \frac{1}{2} \left[ x^2 - y^2 \right].
\]
topic: optimization

Calc I:
Fermat's Thm: If \( f(a) \) is a local extremum, then \( f'(a) = 0 \) or \( f'(a) \) undefined.

- local max
- \( f' = 0 \)
- local min, \( f' \) not defined
- global max
- global min

Extreme Value Thm: If \( f \) is cont. on \([a,b]\), then it attains an abs. max and min on the interval.

Classified critical numbers by:
- 1st DT - used slope
  - works for both types of critical numbers.
- 2nd DT - used concavity
  - only works for \( f' = 0 \) type
  - fails sometimes.

Calc III:

Thm: The Max-Min Thm.
Let \( D \) be a closed and bounded region in \( \mathbb{R}^n \) and let \( f \) be a continuous, scalar field on \( D \). Then \( f \) attains a max and min value at some point \( \vec{a} \) and \( \vec{b} \) in \( D \).
Bak: "closed and bounded"

- closed - region contains all its boundary points

- bounded - \( \| \bar{x} \| < R \) for all \( \bar{x} \in D \)
  i.e. all of D sits inside a sphere of radius R

Our tool: to use the Max-Min Thm

1) look for local extrema in interior of D

We need a way to find and classify critical points.
the 2nd D.T.

2) look for max + min values on the boundary

1st way: parameterize boundary of D and
use a four dimensional Max-Min Thm.

2nd way: method of Lagrange multipliers.

Topic: classifying critical points of the type \( DF(\bar{a}) = 0 \).

Calc I: \( f: \mathbb{R}^1 \to \mathbb{R} \)

By Taylor's Thm,
\[
f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \text{error.}
\]

At critical number, \( f'(a) = 0 \)
$$f(x) = f(a) + f''(a) \frac{(x-a)^3}{3!} + \text{error}$$

$$f(x) - f(a) \approx f''(a) \frac{(x-a)^3}{3!} \text{ if } x \text{ very near } a.$$ 

2nd Deriv Test

- If $f''(a) > 0 \implies f(x) - f(a) > 0$ for all $x$ near $a$
  $\implies f(a)$ is a local min.
- If $f''(a) < 0 \implies f(x) - f(a) < 0$ for all $x$ near $a$.
  $\implies f(a)$ is a local max.
- If $f''(a) = 0 \implies$ test inconclusive

Calc III: 2nd DT for scalar-fn of d variables.

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be $C^2$
Let $(a,b)$ be a critical point of the type $\nabla f(a,b) = \mathbf{0}$.

The 2nd-order Taylor approx. of $f$ at $(a,b)$ is

$$f(x,y) = f(a,b) + f_x(a,b) (x-a) + f_y(a,b) (y-b)$$

$$+ \frac{1}{2} \left[ f_{xx}(a,b) (x-a)^2 + 2f_{xy}(a,b) (x-a)(y-b) + f_{yy}(a,b) (y-b)^2 \right]$$

+ error term

Since $\nabla f(a,b) = \mathbf{0}$ and letting $\Delta x = x-a$, $\Delta y = y-b$,
\[ f_{xx} - f_{xy} = \frac{1}{2} \left[ f_{xx} \Delta x^2 + 2f_{xy} \Delta x \Delta y + f_{yy} \Delta y^2 \right] \]

for all \((x,y)\) near \((a,b)\).

Completing the square:

\[ f(x, y) - f(a, b) \approx \frac{1}{2} \left[ f_{xx} \left( \frac{\Delta x^2}{f_{xx}} \right) + f_{xy} \Delta x \Delta y + \left( \frac{f_{xy}}{f_{xx}} \right)^2 \Delta y^2 \right] \]

\[ = \frac{1}{2} \left[ f_{xx} \left( \frac{\Delta x^2}{f_{xx}} \right) + f_{xy} \Delta x \Delta y + \left( \frac{f_{xy}}{f_{xx}} \right)^2 \Delta y^2 \right] \]

+ we always

+ we always

The sign of \(f_{xx} - f_{xy}\) near \((a,b)\) can be determined:

1. \(f_{xx} > 0\) and \(f_{xx} - f_{xy} > 0\), then
   \[ f(x, y) - f(a, b) > 0 \] for all \((x,y)\) near \((a,b)\),
   - \(f\) has a local min at \((a,b)\).

2. \(f_{xx} < 0\) and \(f_{xx} - f_{xy} > 0\), then
   \[ f(x, y) - f(a, b) < 0 \] for all \((x,y)\) near \((a,b)\),
   - \(f\) has a local max at \((a,b)\).

3. If \(f_{xx} - f_{xy} < 0\), then different \(\Delta x\) and \(\Delta y\)'s result in different signs for \(f(x,y) - f(a,b)\),
   - \(f\) has neither a max or min at \((a,b)\).

4. If \(f_{xx} - f_{xy} = 0\), test inconclusive.
Example: let $f(x,y) = -x^2 + y^2$

Critical points: \[ Df(x,y) = \begin{vmatrix} -2x & 2 \hline 2 & 0 \end{vmatrix} \]
\[ Df(0,0) = \begin{vmatrix} 0 & 0 \hline 0 & 0 \end{vmatrix} \text{ with } (x,y) = (0,0). \]

\[ f_{xx}(x,y) = -2, \]
\[ f_{xy}(x,y) = f_{yx}(x,y) = 0, \]
\[ f_{yy}(x,y) = 0. \]

Note $f_{xx}(0,0) = -2 < 0$ and $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -2(0) - 0^2 = -4 < 0$

This is case 3 above: $(0,0)$ is neither a max or min.

Points $z = -f(x,y)$ is the classic saddle surface.

In general, we call critical points of this type saddle points.
Ex. \( f(x,y) = x^4 + y^4 - 4xy + 1 \)

Has critical points \((0,0), (1,1)\) and \((-1,-1)\).

Classify the critical points.

\[
f_x(x,y) = 4x^3 - 4y, \quad f_y(x,y) = 4y^3 - 4x
g_{xx} = 12x^2, \quad g_{xy} = g_{yx} = -4, \quad g_{yy} = 12y^2.
\]

Note: \( D = g_{xx}g_{yy} - g_{xy}^2 = (12x^2)(12y^2) - (-4)^2 = 144x^2y^2 - 16 \)

\[\text{At } (0,0): \]
\[g_{xx}(0,0) = 12(0)^2 = 0 \quad D(0,0) = 144(0)^2(0)^2 - 16 = -16 < 0 \]

a saddle point at \((0,0)\).

\[\text{At } (1,1): \]
\[g_{xx}(1,1) = 12(1)^2 = 12 > 0 \quad D(1,1) = 144(1)^2(1)^2 - 16 > 0 \]

a local min at \((1,1)\) of \(f(1,1) = -1\).

\[\text{At } (-1,-1): \]
\[g_{xx}(-1,-1) = 12(-1)^2 = 12 > 0 \quad D(-1,-1) = 144(-1)^2(-1)^2 - 16 > 0 \]

a local min at \((-1,-1)\) of \(f(-1,-1) = -1\).
Topic: Constrained Optimization

Ex: Find abs. max and mins of 
\[ f(x, y) = x^2 + y^2 - x - y + 1 \] on the closed disk \( x^2 + y^2 \leq 1 \).

\[ f(x, y) \text{ a poly.} \Rightarrow \text{cont. fn} \]

The disk \( D: x^2 + y^2 \leq 1 \) is closed (contains its boundary curve) and bounded.

By defn. \( \| (x, y) \| \leq 1 \) for all \( (x, y) \) in \( D \).

By Max-Min Thm, \( f(x, y) \) attains a max and a min value on \( D \) (either in interior or on boundary).

For interior max-min, use 2nd D.T.

For exterior (on boundary) need something else.

\[ \nabla f(x, y) = (2x - 1, 2y - 1) \]
\[ \text{If } (x, y) = 0 \quad \text{then } x = \frac{1}{2} \text{ or } y = \frac{1}{2} , \]

\[ f \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{4} . \]

(Recall: don't need to classify this pt!

need to compare it to values of \( f \)
on the boundary.)
disc 2: examining the boundary.

We need to parameterize the boundary
\[ x^2 + y^2 = 1 \Rightarrow r(\theta) = (\cos \theta, \sin \theta), \quad 0 \leq \theta \leq 2\pi. \]

\( f(\theta) \) restricts \( f \) to the boundary curve:
\[ f(\theta) = \cos^2 \theta + \sin^2 \theta - \cos \theta - \sin \theta + 1 \]
\[ = -\cos \theta - \sin \theta + 1 \quad \text{on} \quad 0 \leq \theta \leq 2\pi. \]

To find max + min values, use EVT on \( f(\theta) \).

\[ f(\theta) = \cos \theta - \sin \theta \]
\[ f(\pi) = 0 \quad \text{when} \quad \theta = \frac{\pi}{4} \text{ and } \theta = \frac{3\pi}{4} \]

When \( \theta = \frac{\pi}{4} \), \( r(\theta) = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \) and \( f\left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) = 2 - \sqrt{2} \)

When \( \theta = \frac{3\pi}{4} \), \( r(\theta) = \left( \frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \right) \) and \( f\left( \frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \right) = 2 + \sqrt{2} \)

Also need to check endpoints of \( \theta \)'s interval.

\[ f(0) = f(2\pi) = (1,0) \]
\[ f(1,0) = 1. \]

Combining all info: max of \( f \) is 2 + \( \frac{\sqrt{2}}{2} \) at \( \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \)

min of \( f \) is 1/2 at \( \left( \frac{1}{2}, \frac{1}{2} \right) \)
topic: method of Lagrange multipliers.

A geometric method for locating max and min values of \( f(x) \) subject to a boundary condition \( g(x) = C \).

Ex: Find the extrema of \( f(x,y) = y^2 - x^2 \) subject to \( x^2 + y^2 = 1 \).

\[ z = f(x,y) \text{ a saddle surface} \]

\[ \text{constraint} \ x^2 + y^2 = 1 \]

intersection of constraint and the surface.

Look at level curves:

\[ y^2 - x^2 = C \]
\[ y^2 = x^2 + C \]

level curves of \( f(x,y) \) are tangent to the level curve \( x^2 + y^2 = 1 \) when the max + min occur,
How do we use this?

Recall that gradients are normal to level surfaces.

If the level curves/surfaces are tangent at a point, then the gradients must be parallel.

General picture:

Constraint the surface \( g(\vec{x}) = C \).

\[ \nabla g(\vec{x}) \]

Tangent surfaces: max/min here!

Level surfaces \( f(\vec{x}) = k \).

If \( f(\vec{x}) = k \) and \( g(\vec{x}) = C \) are tangent at \( \vec{x} = \vec{a} \), then \( \nabla f(\vec{a}) \parallel \nabla g(\vec{a}) \).

That is, \( \nabla f(\vec{a}) = \lambda \nabla g(\vec{a}) \) for some scalar \( \lambda \).
Thm: Lagrange Multipliers

Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be \( C^2 \). Let \( S = \{ x \in \mathbb{R}^n \text{ such that } g(x) = 0 \} \). Let \( \bar{a} \) be a point in \( S \) and assume \( \nabla g(\bar{a}) \neq \bar{0} \). Then, if \( f \) has a max or min on \( S \) at \( \bar{a} \), then there is a real number \( \lambda \) such that \( \nabla f(\bar{a}) = \lambda \nabla g(\bar{a}) \).

Def: \( \lambda \) is called a Lagrange multiplier.

Ex: Back to \( f(x,y) = x^2 + y^2 - x - y + 1 \) subject to \( x^2 + y^2 = 1 \).

Here \( g(x,y) = x^2 + y^2 \).

\[ \nabla f(x,y) = \lambda \nabla g(x,y) \]

\[ (2x-1, 2y-1) = \lambda (2x, 2y) \]

or \[ 2x-1 = \lambda 2x \]

\[ 2y-1 = \lambda 2y \]

Solve these eqns any way you can.

1. Always check \( \lambda = 0 \).

\[ \lambda = 0 \Rightarrow 2x-1 = 0 \Rightarrow x = \frac{1}{2} \]

\[ 2y-1 = 0 \Rightarrow y = \frac{1}{2} \]

Note \( \left( \frac{1}{2}, \frac{1}{2} \right) \) is not on the constraint \( x^2 + y^2 = 1 \).
\[ \text{Given: } \frac{dx}{1-x} = \lambda x \Rightarrow \lambda = 1 - \frac{1}{dx} \]

\[ \frac{dy}{1} = \left(1 - \frac{1}{dx}\right) dy \]

\[ \frac{dy}{dx} - 1 = \frac{y}{x} \quad \Rightarrow \quad x = y. \]

Using constraint eqn,
\[ \frac{1}{x^2 + y^2} = 1 \quad \Rightarrow \quad 2x^2 = 1 \text{ or } x = \pm \frac{\sqrt{2}}{2}. \]

Solve pts: \( \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \) and \( \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) \)

Same pts found before,
\[ f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2} \quad \text{min of } f \text{ on } x^2 + y^2 = 1 \]

\[ f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2} \quad \text{max of } f \text{ on } x^2 + y^2 = 1 \]