$$\mathrm{sgn}(\sigma) = \begin{cases} +1 & \#\{i<j \mid \sigma(i)>\sigma(j)\} \text{ even} \\ -1 & \text{odd} \end{cases}$$

$\sigma(1), \ldots, \sigma(k)$

\# pairs ~~in~~ in wrong order

LEMMA 27.2 $\sigma, \mu \in S_k$, ε elem. perm.

$\mathrm{sgn}(\sigma \cdot \varepsilon) = -\mathrm{sgn}(\sigma)$.

su

(a) σ = compos. of k elt. perm. then

$$\operatorname{sgn}(\sigma) = (-1)^k$$

(b) $\operatorname{sgn}(\sigma \circ \mu) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\mu)$

(c) $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$

$$\left[\begin{array}{l} \sigma = \varepsilon_{i_1} \circ \dots \circ \varepsilon_{i_r} \\ \sigma^{-1} = \varepsilon_{i_r} \circ \dots \circ \varepsilon_{i_1} \end{array}\right]$$

(d) $p \neq q$, $\sigma(p) = q$

$\sigma(q) = p$

$\sigma(j) = j \quad \forall j \neq p, q.$

then $\operatorname{sgn}(\sigma) = -1$ [σ <u>transposition</u>]

<u>Proof</u> $\varepsilon = \varepsilon_i$

$\cancel{\sigma(1), \sigma(2), \ldots\ldots}$

$$\sigma(1), \sigma(2), \ldots, \underbrace{\sigma(i), \sigma(i+1)}, \ldots, \sigma(k)$$

$$\sigma \circ \varepsilon = \tau$$

$$\tau(1), \tau(2), \ldots, \tau(i), \tau(i+1), \ldots, \tau(k)$$

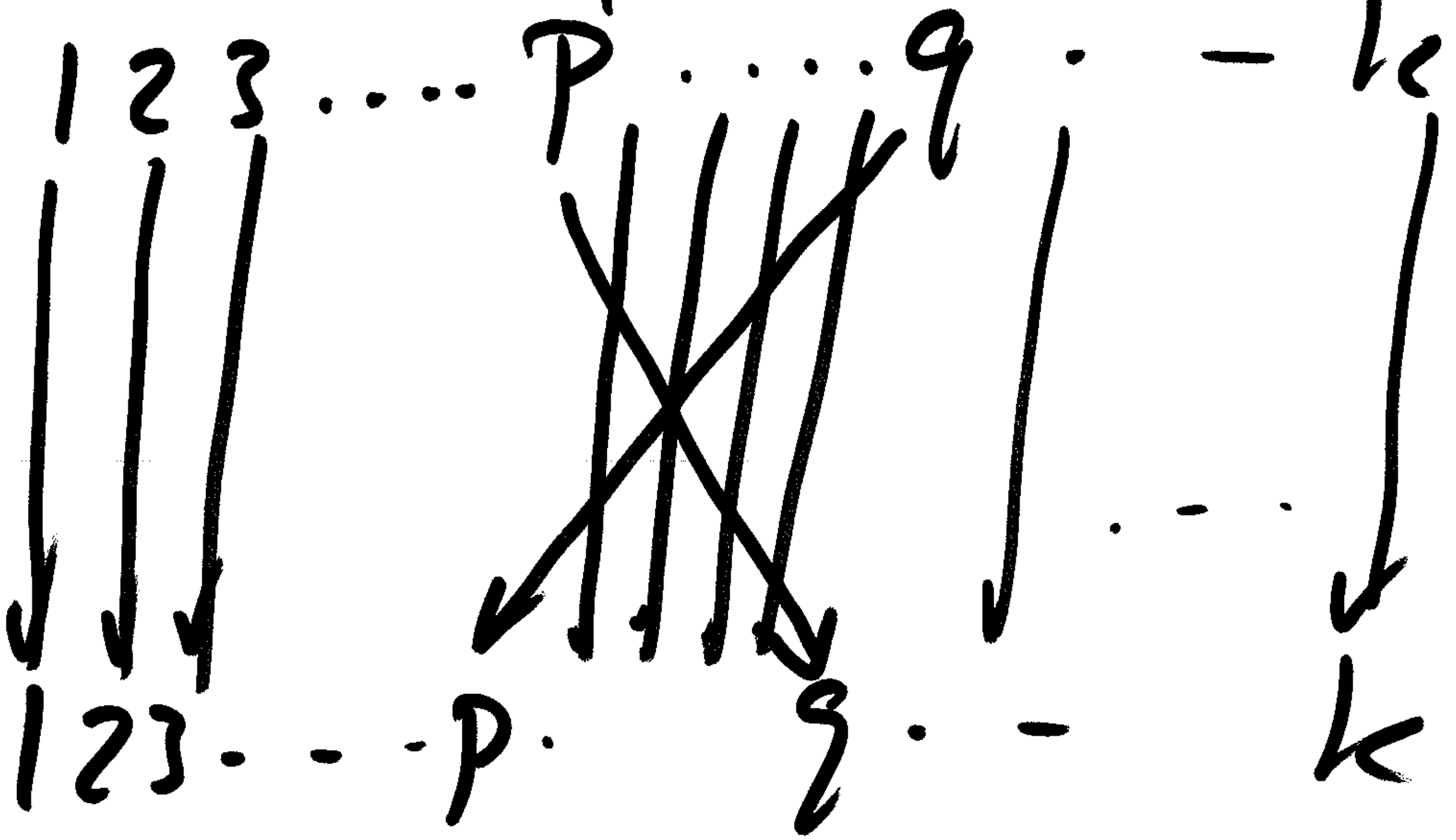
diff in # of inversions is exactly 1. ✓

(a) – (c) already explained

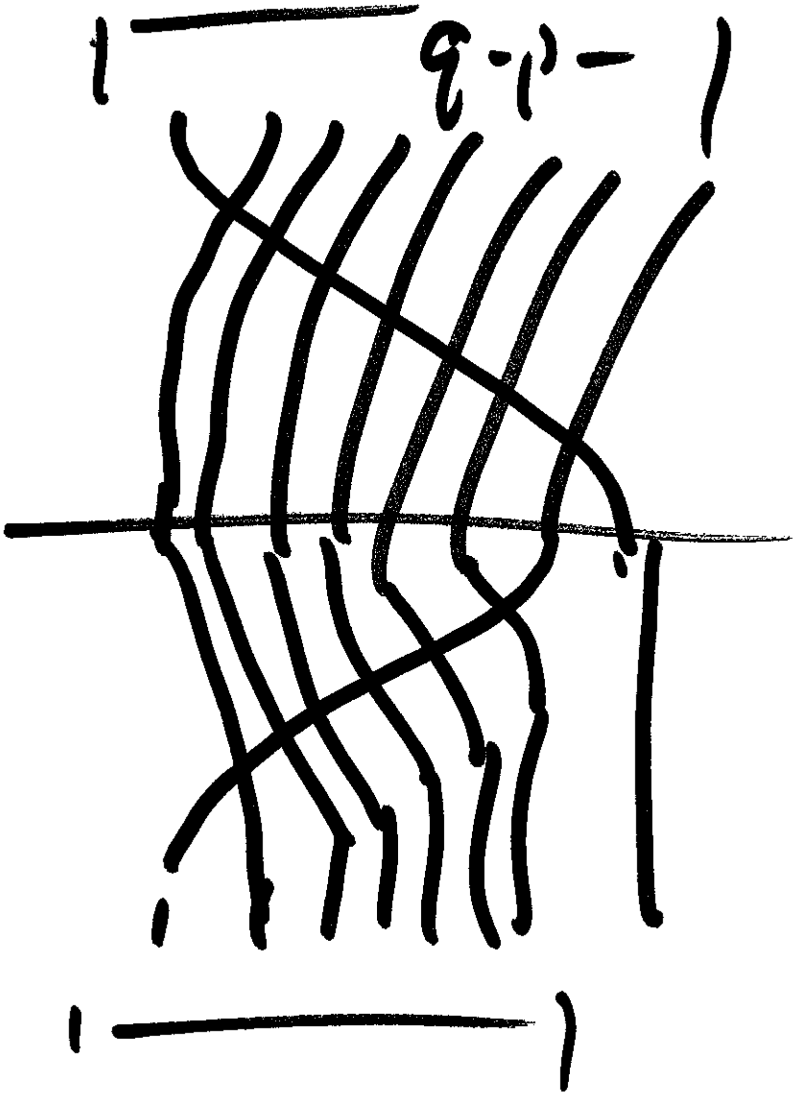
(d) X

1 2 3 · p · · · q · · · k

1 2 3 q · p · · k



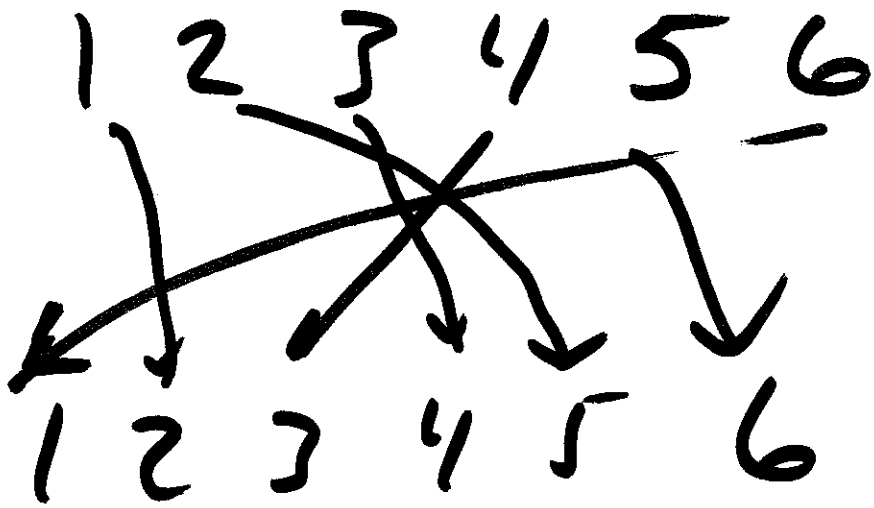
q-p

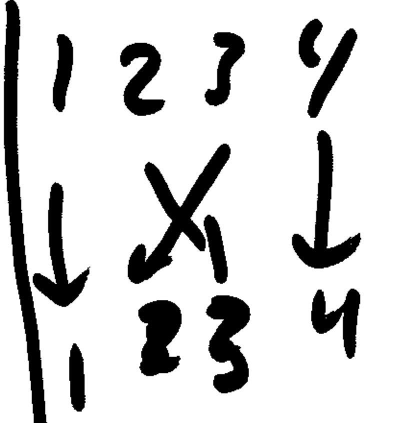


$q-p-1$

$(-1)^{2(q-p)-1} = -1$ odd

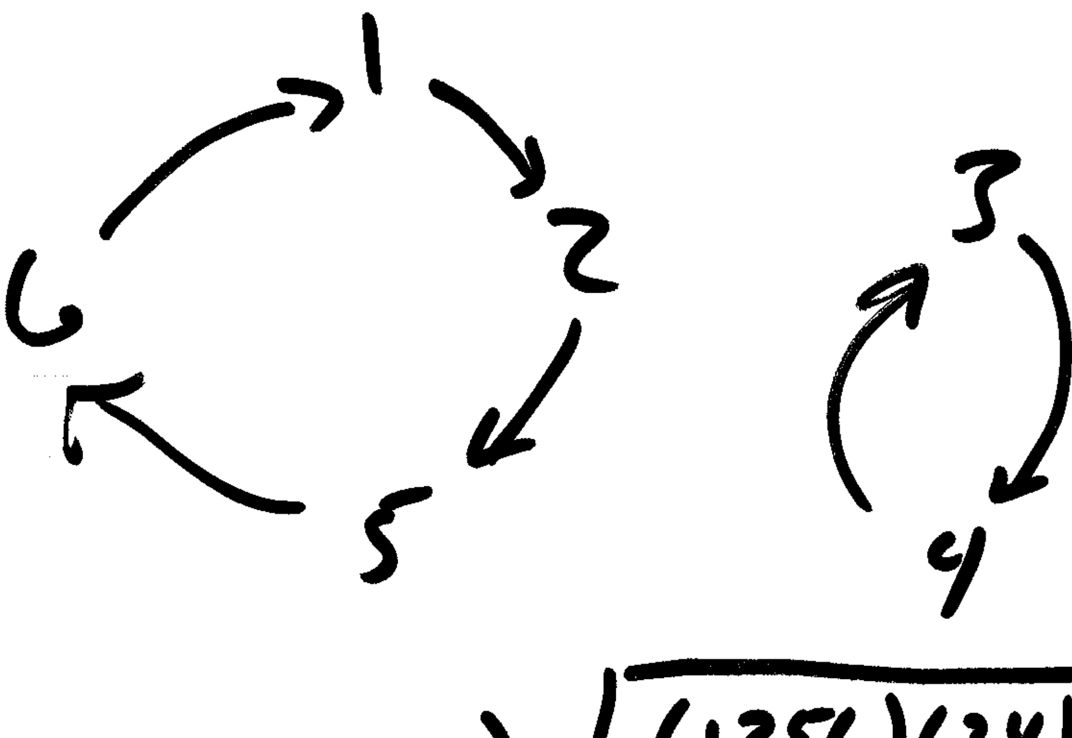
$\square$





$(23)(1)(4)$

$(1\,2\,5\,6)\circ(3\,4)$



k'

$(i\ j)$

$(1256)(34)\circ(123)$

$= (156)(243)$

Given $\sigma \in S_k$

$$\Rightarrow \mathcal{L}^k(V) \longrightarrow \mathcal{L}^k(V)$$

$$T \longmapsto T^\sigma$$

$$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

ε elem. permutation.

$T^\varepsilon = T$ $\forall \varepsilon$, T symmetric

$T^\varepsilon = -T$ $\forall \varepsilon$, T alternating

$A^k(V)$ = alternating tensors

vector subspace of $\mathcal{L}^k(V)$

$k = 1$

$$A^1 = \mathcal{L}^1$$

Lemma 27.3: $T \in \mathcal{L}^k(V)$, $\sigma, \mu \in S_k$

(a) $T \longrightarrow T^\sigma$ is a linear trans.

$$(T^\sigma)^\mu = T^{\sigma\mu}$$

(b) $T \in A^k(V) \iff T^\sigma = \mathrm{sgn}(\sigma)\, T \quad \forall\, \sigma \in S_k$.

(c) $T \in \Lambda^k(V)$, $(v_1, \dots, v_k) \in V^k$

if $v_i = v_j$ for some $i \neq j$

then $T(v_1, \dots, v_k) = 0$

<u>proof</u> (a)-(b) easy

(c) $\sigma \in S_k$

$\sigma(i) = j$

$\sigma(j) = i$

$\sigma(l) = l \quad \forall\, l \neq i, j$

$$\underset{\quad v_i \quad v_j}{T(v_1, \dots, v_k)} = T(v_1, \dots, v_j \dots v_i \dots v_k)$$

$$= T^\sigma(v_1, \dots, v_k)$$

$$= \operatorname{sgn}(\sigma)\, T(v_1, \dots, v_k)$$

$$= -T(v_1, \dots, v_k) \quad \square$$

L 27.4 $a_1, \ldots, a_n$ basis for V

$T, S \in \mathcal{A}^k(V)$

$T = S \iff T(a_{i_1}, \ldots, a_{i_k}) = S(a_{i_1} \ldots a_{i_k})$

$(i_1, \ldots, i_k) \in \{1, \ldots n\}^k$

$i_1 < \ldots < i_k$

$I = (i_1 < \ldots < i_k)$ ascending multiindex

Thrm 27.5

V v.s. $a_1, \ldots, a_n$ a basis $I = (i_1 < \ldots < i_k)$

$\Rightarrow \exists!\ \psi_I \in \mathcal{A}^k(V)$ s.t. $J = (j_1 < \ldots < j_k)$

$$\psi_I(a_{j_1}, \ldots, a_{j_k}) = \begin{cases} 1 & I = J \\ 0 & \text{o.w.} \end{cases}$$

Moreover $\{\psi_I\}$ is a basis for $\mathcal{A}^k(V)$

$$\psi_I = \cancel{\sum_{\sigma \in S_k} \operatorname{sgn}(\phi_I)^\sigma} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)(\phi_I)^\sigma \qquad \begin{matrix} (\phi_I)^\sigma \\ = \phi_{\sigma \circ I} \end{matrix}$$

$\underline{\text{Cor}}$: $\dim(A^k(V))$, $\dim V = n$

$$\dim(A^k(V)) = \begin{cases} \binom{n}{k} & 1 \le k \le n \\ 0 & \text{o.w.} \end{cases}$$

$\quad \| $

$\binom{n}{k}$

$$\boxed{A^0(V) = \mathbb{R}} \quad \text{defin.}$$

$$\binom{n}{k} = 0 \quad \text{if } k > n$$

$$V^0 = \{0\}$$