So, discarding any copies \( \otimes T \) or \( S \times I \) (and in dimensions \( H_2(M,DM; \mathbb{Z}) \)), get \( S'' \vee Y(S''') = X(S'') \), \( \text{[S’’]} \rightarrow [S’’] \)
and \( X_-(S'') = X_-(S') = X_-(S') + X_-(S') = x_0(x_0 + X_0) \).

\( \therefore \), \( x_0(x_0 + X_0) \leq X_-(S''') = X_0(x_0 + X_0) \).

Prove the claim 17.

For every \( x, y \in H_2(M,DM; \mathbb{Z}) \), \( n, m \in \mathbb{Z}_+ \) we have

\[ x(nx + my) = x(nx) + mx(y) = n x(x) + mx(y). \quad (\star 2) \]

Next, extend \( x \) to \( H_2(M,DM; \mathbb{Q}) \) so that \( \forall x \in H_2(M,DM; \mathbb{Z}) \)

\[ x(r \alpha) = r x(\alpha), \]

by \( (\star 1) \), independent choice of \( r \alpha \):

\[ x\left(\frac{r}{k} (x) - \frac{r}{k} (x)\right) = \frac{r}{k} x(x) = \frac{r}{k} x \alpha \]

\[ = r x(\alpha). \]

\( \Rightarrow \) Well defined on \( H_2(M,DM; \mathbb{Q}) \). By \( (\star 2) \), it

is convex: \( \forall \alpha, \beta \in H_2(M,DM; \mathbb{Z}) \)

\[ x((1-t)\alpha + t\beta) = x(\frac{t}{k} (1-t) \alpha + t \beta) = \frac{t}{k} x(k(1-t) \alpha + k \beta) \]

\[ \leq \frac{1}{k} (k(1-t)x + k \beta) = (1-t)x(x) + tx(\beta) \]

\[ \Rightarrow \] \( x \) has a unique continuous extension to \( H_2(M,DM) \), \( \forall \)

\[ x(x + \beta) \leq x(x) + x(\beta) \] \( \forall x, \beta \in H_2(M,DM) \),

and \( x(tx) = tx(x) \) \( \forall t \in \mathbb{R}_+, x \in H_2(M,DM) \),

\( \Rightarrow \) \( x(x) = 0 \Rightarrow x = 0 \) (by construction, \( x(x) = x(x) \) \( \forall x \in H_2(M,DM; \mathbb{Z}) \) heuristically).
The norm \( \| \cdot \| \) is very different than most norms generally considered — never comes from an inner product, for example.

In fact, if we let \( \mathcal{B} \) be the unit ball in \( H^2(M, \mathbb{M}) \), and \( \mathcal{B}^* \) unit ball for dual norm \( H^2(M, \mathbb{M}) \), then here

**Theorem 4.3**

\( \mathcal{B}^* \) is a polyhedron in \( H^2(M, \mathbb{M}) \) whose vertices are lattice points, \( \pm \beta_{i,k} \), and the unit ball \( \mathcal{B}_x \) is a polyhedron defined by

\[
\mathcal{B}_x = \{ x \mid 1^T x \beta_i, |x_i \beta_i| \leq 1 \text{ for } 1 \leq i \leq k \}
\]

If \( \mathbb{M} = \mathbb{R} \), \( \beta_i \) are even. (\( \beta_i = 2 \phi_i \))
proof of theorem 4.3. This has nothing to do with $\mathbb{F}_2$-

This is a fact about norms taking $\mathbb{Z}$-values on an $\mathbb{Z}$-lattice in $\mathbb{R}^n$.

Let $B_x$ be unit ball.

Claim: $\forall a \in H_2(M, M; \mathbb{Z}), \exists$ an integral linear function:

$L_a: H_2(M, M; \mathbb{Z}) \rightarrow \mathbb{Z}$ (so $L_a(H_2(M, M; \mathbb{Z})) \subseteq \mathbb{Z}$) \wedge

$L_a(a) = x(a)$ and $B_x \subseteq L^{-1}_a(\{0\})

Note that $L$ is an integral elt of

$H^2(M, M; \mathbb{Z}) \cong (H_2(M, M; \mathbb{Z}))^*$

Thus, since rays through $H_2(M, M; \mathbb{Z})$ are dense in $H_2(M, M)$

we have

$B_x = \cap L^{-1}_a((0, 1])_{a \in H_2(M, M; \mathbb{Z})} \wedge x^*(L_a) = \sup_{y \in B_x} L_a(y) = 1

So, $\{L_a\}_{a \in H_2(M, M; \mathbb{Z})}$ all lie in $B_{x^*}$ which is open, and

hence this is a finite set and $B_x$ is the required polyhedron.

proof of claim when $H_2(M, M; \mathbb{Z}) \cong \mathbb{Z}^2$. See Thm. 4.3 dum = $\mathbb{Z}$,
giving good idea.

Pick an isomorphism $H_2(M, M; \mathbb{Z}) \cong \mathbb{Z}^2$ w/ $H_2(M, M; \mathbb{Z}) \cong \mathbb{Z}^2$ and

$a = (0, 1), \ b = (1, 0)$, write $x(a) = m$ and $x(ka + kb) = x(k1) = nk, \ 
\forall k \in \mathbb{Z}^>0$. Note $n, k \in \mathbb{Z}, \ \forall k.

Let $L_k = \mathbb{Z}$ linear function agreeing w/ $x$ on $a$ and $ka + kb$
\( L_k^{-1}(1) \) meets lines through \( a \) and \( k \) at \( \partial B_2 \).

Here are the pts \( t_m(0,1) \) and \( t_n(1,k) \).

The slope of \( L_m^{-1}(1) = \frac{t_n - t_m}{m} = k - \frac{n}{m} = \frac{k \cdot m - n}{m} \).

These slopes are non-decreasing by convexity.

And bounded above by \( \frac{k}{m} = \frac{n_0}{m} \) by convexity.

Since we have a bound on denominators, \( L_k^{-1}(1) = L_{k'}^{-1}(1) \) \( \forall j \geq k \)

for some \( k \) sufficiently large. By convexity, in fact, once \( L_k^{-1}(1) \) and \( L_{k'}^{-1}(1) \) are the same for some \( k \), then

\[ \mathbb{P}_+(D,1) + \mathbb{P}_+(1,k), \quad x = L_k \]

Setting \( L_a = L_k \) completes the proof. \( \square \)
Example. Let $L \subset S^3$ be a link of $k$-components.

This is an embedding of
\[ \bigsqcup_{i=1}^k S^1 \subset S^3. \]

Let $M = S^3 \setminus N(L)$ = exterior of $L$

3-manifold $\partial M \approx \mathbb{R}^2$.

If $L = L_1 \cup \ldots \cup L_k$, then $\text{Excision}$. Why? Why not $\text{L.E.}$, sq. sq. rel. hand,

\[ H_2(M, \partial M) \cong H_2(S^3, N(L)) \cong H_2(S^3, L) \cong H_1(L) = H_1(\mathbb{R}^2) \cong \mathbb{R}^k. \]

and similarly, $L_i$ get an integral $\pi_i$ for this bundle. $\partial L_i \approx L_i$.

To more directly relate these $L_i$ to an ell of $\partial(M \cup M)$ take any surface $\approx S^2$ whose boundary is $L_i$, then $S_i \cap M$ (oriented $S_i$)

denoted $S_i$, give a unique representation by the class.

\[ \text{Ex: } W = L_1 \cup L_2 \subset S^3 \text{ the Whitehead link. } \]

Fact: $\partial W$ is treed, abraded, i.e. ~ later maybe...

\[ L_i = 2D_i, \ D_i \text{ a disk. } \]

so $S_i \cap M$ is disk w/ 2 holes (or paired punctures).

\[ \chi_-(S_i) = -1 \]

So, $1 \leq \chi(L_i) \leq \chi_-(S_i) = 1$

$\partial L_1 + L_2 = \partial S_i$

\[ \chi_-(S) = 2 \]
Since \( S' \) must have at least 2 boundary components, we have

\[ X_-(S') \leq -2g - 2 = -2g. \]

If \( g = 0 \), then \( S' \) is annulled after having \( \geq 4 \) boundary components, so \( X_-(S') = -2 \). \( \therefore \), \( X(S') = X(L_1 + L_2) \), so

\[ \frac{2}{2} \]

can draw 2 faces. Similar argument (or symmetric).

By a similar calculation (see Theorem), one can compute

\[ \frac{2}{2} \] 

for the Barrowman rings.
Fibered 3-manifolds

The Thurston norm is particularly useful in studying 3-manifolds that fiber over $S^1$: $S \to M \to S^1$.

For $z \in S^1$, $\pi^{-1}(z) \cong S$ and let $N(z) \cong \mathbb{R}/1\mathbb{Z}$ we have

$$
\pi^{-1}(N(z)) \cong S \times \mathbb{R}/1\mathbb{Z}, \quad L.E.S., \quad \ker \gamma^3 \cong \mathbb{Z}.
$$

So, $M = M_\phi$ for some $\phi : S \to S$. (Recall $M$ is irreducible.)

$\phi$ is called the monodromy and is well-defined up to isotopy.

**Proposition 4.14**: Suppose $M$ is fibered by fiber $S$, $\chi(S) < 0$ and monodromy $\phi$. Then $M$ is abelian iff $\phi$ is reducible.

- $\phi$ is reducible if $\exists C \subset S$ a union of non-null tori, non 3-balls, $S^2 \cup_\phi S^2$ is isotopic to $C$, or $\phi$ is irreducible.
- $\phi$ is periodic if $\phi^k \equiv \text{id}$ for some $k > 0$.

**Proof**: If $\phi$ is reducible, let $C$ be as in (a), apply isotopy of $\phi$, so $\phi_0(C) = C$. Then $C \times S^1 \cong C \times [0,1] \to M_{\phi_0} \cong M_\phi$ projects to a union of noncompressible, which are not irreducible, $S^2 \cup_\phi S^2$, these lift to a union of annuli $C \times R^2 \cong C \times [0,1]$. 


Conversely, suppose \( \phi \) is irreducible and \( T \) an irreducible tranverse \( T \subset M \).

By step 4, make \( T \) closed and assume \( |S^1| \) is minimal.

If \( T \) is not irreducible, \( \text{M-intersect} \) no disk comps of \( T \) or \( S^1/T \).

Since \( X(T) = 0 \), all comp of \( T \) or \( S^1/T \).

No disks in \( S^1/T \Rightarrow S^1/T \text{ is a union of non-annular } S^1 \text{ s.}

Annuli \( T/S \) determine an isometry from \( C \to \phi(C) \) (requires some work).

\( \text{so } C \text{ is } T \text{-parallel} \Rightarrow T \text{ is } T \text{-parallel.}

\( \text{so no counts like this by minimality.} \quad \square \)

**Ex**

Conclude irreducible homomorphism.

Euclidean metric w/ singular 3 from vertex at \( z_0 \).

Note linear maps \( T_1 = (0, 1) \) and \( T_2 = (1, 0) \).

Define maps \( d\phi \) which are "affine" on \( S^1(\pi) \) and have derivatives \( (\phi') \) \& \((\phi')' \) w.r.t. basis \((e_1, e_1')\)

for \( T \), at all \( z \in S^1 \).

Let \( G = \langle T_1, T_2 \rangle \) the group of homom \( ( \text{diffs of } S^1(\pi) ) \).

Get \( D(G) \to \mathbb{SL}_2 \) by taking deriv. w.r.t. \((e_1, e_1')\), thus a homomorphism by chain rule: \( D(fh) = Df \circ D(h) \).

\[ D(G) = \langle (0, 1), (1, 0) \rangle < \mathbb{SL}_2 \]
Support $\varphi \in G$ is any elt w/ $|\text{tr}(D\varphi)| > 2$

(true for any elt not conj to a power of $T_{1, T_2}$ or $T_{1, T_3}$)

(e.g. $T_1 \cdot T_2$, $2 \times 2$)

By square, we can assume $\text{tr}(D\varphi) > 2$. The $D\varphi$ has 2 eigenvalues $2, \frac{1}{2}$ w/ $2 > 1$. Let $U$, $V$ be eigenvectors. These define parallel vector fields on $S^2(S^3)$ and integrating these gives 2 solutions of $S^2(S^3)$ by straight lines, $\gamma_+$, $\gamma_-$.

Note of Stretch: all lines $\gamma_+$ $\gamma_-$ by $A$ and contract all lines $\gamma_+$ $\gamma_-$ by $\frac{1}{A}$.

Given any curve $\omega$ on $S^2$, apply stretching to make a geodesic.

If $\omega \subset S^2(S^3)$, the $\omega$ is a closed geodesic. Otherwise $\omega$ is a concatenation of straight lines from $\omega$ to itself. In any case $\varphi^\omega(\omega^\omega)$ is the geodesic rep of $\varphi(\omega)$ (i.e., $\varphi(\omega)$ is metric is non-posit).

and length $\|\varphi^\omega(\omega^\omega)\| = \infty$: indeed, $\varphi^\omega(\omega^\omega)$ looks more and more like $\omega^\omega$, $\gamma_\pm$. In particular, $\varphi$ cannot be reducible.

This structure: Sing. curve metric (rpc) $\varphi$ affine preserves sides $\gamma_+, \gamma_-$, stretches $\gamma_+$ by $2 > 1$ contract $\gamma_-$ by $\frac{1}{2}$, makes $\varphi$ a pseudo-anosov homomorphism.

Theorem (Thurston) If $\varphi$ is ended, non-periodic, then $\varphi \in \mathcal{P}_0$, pseudo-Anosov.