We won't prove this (or use it) but it's good to know.

Back to $S \to M = M_{\nu} \to S'$. Note $[S] \in H_2(M, \mathbb{Z})$ is nonzero.

Proposition 4.5: Let $\alpha \in \text{Ann } H_2(M, \mathbb{Z})$, $\alpha$ is represented by a fiber $S'$ for some fibration $S' \to M \to S'$.

**Proof:** Let $\nu$ be the volume form on $S'$, with $\int_{S'} \nu = 1$.

Then $[\nu] = \text{dual to } [\pi^*(\nu)] = \{\alpha\} \in H^1_{dr}(M) \cong H^1(M)$.

Note that $\nu/\alpha$ is a fibrating: $\pi$ is a submersion. So $H^1_{dr}(M)$ vanishes. Pick a basis for $H^1_{dr}(M)$ and represent these classes by closed 1-forms $\eta_0, \eta_1, \ldots, \eta_k$. The set $\alpha = 0$ on $H^1_{dr}(M)$ by

$$N_\alpha(\eta_0) = \{ \sum_{i=1}^k \xi_i \eta_i : \xi_i \in H^1_{dr}(M) \mid \sum_{i=1}^k |x_i| < 1 \text{ for all } x_i \}$$

Set $N_\alpha(\eta_0) = \{ \eta_0 + \sum_{i=1}^k \xi_i \eta_i : \xi_i \in H^1_{dr}(M) \mid \sum_{i=1}^k |x_i| < 1 \text{ for all } x_i \}$. Note that

\[ N_\alpha(\eta_0) = \text{Ann } [\alpha] \in H^1_{dr}(M) \cong H^1(M) \text{ is a submanifold of } M, \text{ for some sufficiently small } \alpha.

Via $H^1_{dr}(M) \to H^1(M, \mathbb{Z})$ integral classes $\eta \in H^1(M, \mathbb{Z})$ are then integral points.

Put $N_\alpha(\eta_0) \to N_\alpha([S])$ for any such $\eta_0$ and $M \to S'$ given by $\eta = [\eta]$. This is submersive.
Let $f: \tilde{M} \cong S^2 \times \mathbb{R} \rightarrow S$ be the projection map. Let $w \in \Omega^2(S)$ be an odd form with $\int_S w = 1$. The degree $\deg(f|_S')$ is given by $\int_{S'} w = \int_{S} f^* w = 1$ since $f|_S: S \rightarrow S'$ is a proper map. Therefore, $\deg(f|_S') = 0$ or $1$.
Since \( \hat{f} = g \mid_{S'} : S' \to S \) is degree 1 and since 
\[
\hat{f}^* \text{ is } \gamma = 1, \text{ on } H^1_{an}(S) (\text{and } H^1_{dR}(S, \mathbb{Q})).
\]
(If \( S' \) has mult curls, \( \phi' \) has deg \( 0 \) sum curl \( S' \), ...)

\[\Rightarrow \chi(S') \geq \chi(S), \text{ so } S \text{ is minimizing.} \]

Consider tangent plane to fibers \( S \to M \to S' \), the (relative) divisor class \( e \in H^2(M, \mathbb{Q}) \) is the obstruction to finding a section — compute it by picking a section.

\[\sigma : M \to \mathbb{Z}, \sigma \equiv 0 \mod 2 \text{ (and rank points on } \mathbb{P}(M)).\]

\[e(\sigma) - [\delta^{-1}(\mathbb{Z})] \in H_1(M)\]

where \(|\langle e(\sigma); [S] \rangle| = |\chi(S)| = \chi(S) = \chi([S])\).

**Theorem 4.7** \( \chi(e(\sigma)) = 1 \) and \( e(\sigma) \) defines a 2-form \( F \in \Omega^2_{\mathbb{R}(M)} \); if \( [S] \in \mathbb{R}^+ \cap [S] \),

\[\langle e(\sigma); [S] \rangle \text{ is } \chi(S), \text{ a neighborhood of } [S].\]

**Proof:** Proof \( 9 \). \( 3 \Rightarrow \) a neighborhood \( N_\varepsilon([S]) = N_\varepsilon(\eta_0), \varepsilon = \varepsilon([S]) \)

\( [\eta] \in N_\varepsilon(\eta_0) \) of the form \( \eta = \eta_0 + \sum \varepsilon_i \eta_i \mid x_i \leq \varepsilon \)

\( \eta_x = [\ker(\eta)] \) is the \( 2 \)-plane field on \( M, x \in x \), \( \forall x \in (\mathbb{R}) \) \( \langle x \rangle \leq 3 \)

\[\varepsilon([S]) = e(\sigma), \text{ if } \eta_x \text{ is } \mathbb{R}^2 \text{, hence dual to } \varepsilon \text{, where } \chi = \chi([S]), \varepsilon = \chi([S]) \]

so \( \chi([S]) = |\varepsilon([S])| \) holds in dense set, so one manifold...
Theorem 9.8: There are a and \( F_1, \ldots, F_k \) \( B \)-top dimensional classes \( A \) such that the fiberwise \( C^2 \)-bundle over \( \mathbb{M'} \) is exactly
\[
H_2(\mathbb{M}, \mathbb{Z}) \otimes \left( \bigoplus_i \mathbb{R}_e^{F_i} \right)
\]
In fact, \( \bigoplus_i \mathbb{R}_e^{F_i} \) is properly null homotopic to \( H(\mathbb{M}) \) representable by nontrivial closed 1-forms.

By 4.7, the norm \( \rho \) restricted to \( \mathbb{R}_e^{F_i} \) is given by \( e(\xi) \), where \( \xi = 2 \)-plane field tangent to the fibers \( S \to \mathbb{M} \to \mathbb{S} \) of \( S \subseteq \mathbb{R}^{e} \).

To prove this, we will need to be able to isotope any curve \( S \) in \( M \) to a nice position with the foliation by fibers, and any general foliation carries an orientation.

Theorem 14.9: Suppose \( F \) is a \( 3 \)-Ly oriented taut codim 1 foliation on \( M \) of \( E(M) \). Then, given a properly embedded incompressible surface \( S \subseteq M \) of \( F \) \( \Psi \) is a leaf of FF for every component.

Suppose \( S \) can be isotoped so that either \( S \) or a leaf of \( F \) in \( S \) has only isolated saddle tangencies.

For a proof, see Candel, Conlon, Foliations II, §9.5, or Candel, Foliations and the geometry of 3-manifolds, §5.

We explain the times that are isolated saddle tangencies.
If is that if through every leaf there passes an closed curve \( \gamma \) or \( \gamma \) embedded are which is traversed by \( \gamma \).

E.g., foliate \( \gamma \) by fibres of homomorphism \( S \to M \to S' \), all leaves are \( \gamma \)-symplectic.

We can restrict \( \gamma \) to \( S \to S' = \gamma \cap S \), foliated by \( \gamma \)-symplectic of \( \gamma \) provided \( \gamma \) is \( \gamma \)-symplectic. If \( \gamma \) has finitely many exceptional pts, then these points become singularities of \( \gamma \). Saying \( \gamma \) has only isolated saddle-tangencies means that \( \gamma \) is at all but finitely many pts, and at these pts, the singularities is a saddle.

More exactly, the foliation \( \gamma \) is locally described by \( \gamma \)-sub. of a submersions \( \gamma \to \mathbb{R} \), and near pt where \( \gamma \) is \( \gamma \)-transvers to \( S \), we require \( \gamma \)-sub. \( \gamma \to \mathbb{R} \) to be a Morse function \( \gamma \)-index 1.

E.g., if \( \gamma \) is the foliation - defined by \( \eta \in \mathfrak{X} \gamma = \mathbb{R} \cdot \chi (\mathbb{R}) \subset \mathfrak{X} \gamma \) on \( \gamma \), \( \eta \) a nowhere vanishes closed 1-form, then locally \( \eta = \mathbb{R} \), w/ \( \gamma \)-subersion defining a splitting \( \gamma = \gamma \). Equiv., the tangent space to \( \gamma \) is \( \ker (\eta) = \ker (df) \) which in fact defines \( \gamma \). The foliation \( \gamma \) is defined by \( \eta \)-sub. of \( \gamma \)-symplectic.

In general, \( \eta \gamma \) is \( \eta \gamma \)-symplectic as well as above.
Recap \( S_0 \to M \to S' \) fibration \( [S_0] = [\gamma_0] \in H^1_{dR}(M) \)

\( \gamma \) induces vanishing closed form.

\( \exists \quad U = R^*_+ N_c(\gamma_0) \subset H^1_{dR}(M) \) w/ every elt of \( U \)

rep'd by a closed 1-form vanishing 1-form \( \gamma \), \( \gamma |_{\partial U} \), and so

all \( \gamma \) on \( U \cap H^1_c(M, \mathbb{Z})^{\text{tame}} \) represented by a fiber.

Any such \( \gamma \), w/ \( \gamma |_{\partial U} \) defines codim 1 foliation \( F = \mathbb{R}^*_+ \), transversely

orient'd by \( \gamma \). Tangent plane field \( \tau = \tau_{\gamma} = \tau_{{\gamma}_0} \). For \( \gamma = \gamma_0 \), leaves on

fibers (similarly for any integral class \( U \)), all isomorphic 2-plane bundles over \( M \).

Have relative Euler class \( e(U) \in H^2(M, \mathbb{Z}) \), relative to inertia plus normal,

depends only on \( U \), not on \( \gamma \). Furthermore, \( x^*(e(U)) = 1 \) and

\( \exists \ F \) a top dim'lslice \( \tau = \mathbb{R}^*_+ \)

\[ x|_{R^*_+ F} = 1 \langle e(U), \tau > 1 \]

In particular, \( U \subset R^*_+ F \).

Goal: show every elt of \( R^*_+ F \cap H^1_c(M, \mathbb{Z})^{\text{tame}} \) is represented

by a fiber in a fibration over \( S' \).

Given any \( F \), e.g. \( F = F_\gamma \), and an imcompressible, prop embedded

\( S \subset M \) can isotop \( S \) so either \( S \) is a leaf of \( F \) or \( S \) \( \neq F \) except

at finitely many singular pts, and at these pts, has saddle

\[ S \]

In \( M \), these are exactly the pts \( m \)

of \( S \) where \( T_m(S) = \mathbb{R}^m \), when \( \gamma = \gamma_0 \).
In this situation we want to compute \( \langle e(\tau), [S] \rangle \), which we do via an index formula, as follows.

Special case, \( S \) a leaf of \( \mathcal{G} \) \( \Rightarrow \langle e(\tau), [S] \rangle = \langle e(\tau(S)), [S] \rangle = X(S) \),

If \( S \) not a leaf, consider \( \mathcal{G}_S \), sing. foliation on \( S \).

Call a singular point positive if \( T_m(S) = 2_m \) as oriented vs.,

and negative if \( T_m(S) = -2_m \).

Set \( I_p = \sum_{m a} \text{Ind}_{m a}(m) = \# \text{ pos. sing. pts of } \mathcal{G} \)

\( I_n = \sum_{m a} \text{Ind}_{m a}(m) = \# \text{ neg. sing. pts of } \mathcal{G} \).

**Proposition 4.10**. \( S, \mathcal{G}, M \) as above, \( S \) not a leaf of \( \mathcal{G} \). Then

(a) \( \langle e(\tau), [S] \rangle = I_p - I_n \)

(b) \( X(S) = I_p + I_n \)

**proof:**

We pick a section \( \sigma : S \to \mathcal{G} \), \( \sigma(\tau(S)) \neq T(S) \).

Construct \( \sigma = \sigma(m) \in \mathcal{T}_{m}S \cup T(S) \).

\( \sigma_0 = \text{null normal} \), then \( \langle e(\tau), [S] \rangle = \langle \mathcal{E}(\tau(S), \sigma(\tau(S)) \rangle \) alg. nd. \# \text{ in } T(S).

\( \sigma(m) \neq 0 \) \( m \) is singular.
In particular, \( \tau \) is a vector field on \( S \), which is \( \partial / \partial z \) on \( \mathbb{C} \). Thus,

\[ I_p + I_N = -\# \partial \delta \text{ sing } \partial / \partial z = \sum_{\delta \in \mathcal{P}} m_{\delta}(\delta) = X(S) \text{ by Poincaré-Hopf.} \]

On the other hand, at an irregular pt on \( T_n(S) = \partial T \) and \( \tau \) is also transverse \( \mathbb{R}(S) \), and, sign \# \partial \text{ intersection pt } \partial (\delta) \in \mathcal{P}(S)^{\mathbb{R}(S)} \text{ is same as sign } \partial \text{ when } \mathcal{P}(S)^{\mathbb{R}(S)} \text{ coincide.}

So,

\[ \langle \epsilon(e), [S] \rangle = I_p - I_N \text{, as required.} \]

The next theorem replies -1.4.

**Theorem 4.11** \( S \rightarrow M-3 \), \( F \) face of \( B_x \) defined by \( e(t) \) w/\n
\[ \langle e(t), \cdot \rangle \mid_{R_f} = \alpha \]

where  \( \alpha \) a 3-plane field \( S \) foliated by \( F \).

Then every odd \( R_f \) is represented in \( H^1(M) \) by a nowhere vanishing 1-form \( \eta \).

**Proof** Let \( \eta_0 \) be a nowhere vanishing 1-form dual to \( S \).

In \( H^1(M) \), and \( \eta_0 \) foliated by fibers. Given any integral class

\[ S \in \text{ a leaf } \eta_0 \text{ or } S \# S, \text{ explain why any saddle tangent.} \]

If \( S \) is a leaf, then \( [S] = [S] \), so dual \( \eta_0 \) \( S \) is rep by \( \eta_0 \), nowhere vanishing.

If \( S \) not a leaf, then Prop 4.10 \( \Rightarrow I_p - I_N = \langle e(t), [S] \rangle \)

and \( I_p + I_N = X(S) \). Since \( [S] \in R_f \), \( \langle e(t), [S] \rangle = \omega(S) = X(S) \).
So, \( I_N = 0 \) or \( I_P = 0 \) in fact, \( I_N = 0 \) since \( c(\tau, 0) > 0 \) on \( P' \cdot F \).

One can choose a weaker \( E \) on \( M \) along \( S \), which is transverse to balls of \( 2S \). Extend this to \( N(s) \), still \( m \), and change to \( \mathbb{E} \) (to \( E \)) of \( N(s) \). Also, \( E \) is not partial on \( N(s) \).

Let \( \eta \) be a smooth 1-form dual to \( S \), so \( \eta \) is supported on \( N(s) \).

Let \( \eta \) be a smooth 1-form dual to \( S \), so \( \eta \) is supported on \( N(s) \).

\[ 0 < (\eta, \langle E \rangle) = \eta \cdot \langle E \rangle, \quad \text{also} \quad \eta \cdot \langle E \rangle > 0 \]

\( \forall \tau \in (0, 1) \), consider

\[ \eta_\tau = \tau \eta + (1 - \tau) \eta_0 \quad \text{so} \quad \eta_\tau = \eta_0 \]

Note: \( \eta_\tau \mid_{M \setminus N(s)} = \eta_0 \mid_{M \setminus N(s)} \) nowhere vanishes.

\[ \forall \tau \in N(s), \quad (\eta_\tau, \langle E \rangle)_m = \tau (\eta, \langle E \rangle)_m + (1 - \tau) \eta_0 \langle E \rangle)_m > 0 \]

So, \( \eta \) nowhere vanishes on \( \mathbb{R}_+ \cdot E \cdot \eta \). Let \( \eta \), and (57w) another integral class in \( \mathbb{R}_+ \cdot F \), can get \( \eta \) from an convex hull \( A \) all integral pts \( \mathbb{P} \) of \( \mathbb{R}_+ \cdot F \) interior \( A \)

represented by nowhere vanishing 1-form, but this all of \( \mathbb{R}_+ \cdot F \).