Section6.1 Exercise6:

a) $det(A - xI) = det \begin{pmatrix} 3-x & 2 \\ 4 & 1-x \end{pmatrix} = (x-5)(x+1)$. Thus eigenvalues are $5, -1$.

$\boxed{\lambda_1 = 5}$ We want to solve $\begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$. It's easy to see that $x = y$, hence the eigenspace is $V_1 = \{(x,x)^T\} = span\{(1,1)^T\}$.

$\boxed{\lambda_2 = -1}$ We want to solve $\begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$. It's easy to see that $2x + y = 0$, hence the eigenspace is $V_1 = \{(x,-2x)^T\} = span\{(1,-2)^T\}$.

f) Clearly, the only eigenvalue is 0. Solve

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0},$$

we get $y = z = 0$. Hence the eigenspace is $V = span\{(1,0,0)^T\}$.

h)

$$\begin{vmatrix} 1-x & 2 & 1 \\ 0 & 3-x & 1 \\ 0 & 5 & -1-x \end{vmatrix} = \begin{vmatrix} 4-x & 2 & 1 \\ 4-x & 3-x & 1 \\ 4-x & 5 & -1-x \end{vmatrix} = (4-x) \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3-x & 1 \\ 1 & 5 & -1-x \end{vmatrix}$$

$$= (4-x) \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1-x & 1 \\ 0 & 3 & -2-x \end{vmatrix} = (4-x)(1-x)(-2-x).$$

Thus eigenvalues are $4, 1, -2$.

$\boxed{\lambda_1 = 4}$

$$\begin{pmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}.$$

From the second and third row we have $y = z$. Substituting into the first row we have $0 = -3x + 2y + z = -3x + 3y$ and hence $x = y = z$. The eigenspace is $V_1 = \{(x,x,x)\} = span\{(1,1,1)^T\}$.

$\boxed{\lambda_2 = 1}$

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}.$$

Spelling out this equation we get $2y + z = 0 = 5y - 2z$ and hence $y = z = 0$. The eigenspace is $V_2 = \{(x,0,0)^T\}$.

$\boxed{\lambda_3 = -2}$

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}.$$

From the second row we have $z = -5y$, substituting into the first row we have $0 = 3x + 2y + z = 3x - 3y$ and hence $x = y$. Thus the eigenspace is $V_3 = \{(x, x, -5x)^T\}$. $\square$

Section6.1 Exercise6:
Since λ is an eigenvalue of A, there is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. Applying A on both sides, we have $A^2\vec{v} = A(\lambda\vec{v}) = \lambda^2\vec{v}$. Now we are given $A^2 = A$, thus we have $A^2\vec{v} = A\vec{v}$. This implies $\lambda^2\vec{v} = \lambda\vec{v}$. Since $\vec{v}$ is a nonzero vector, we must have $\lambda^2 = \lambda$. This gives $\lambda = 0, 1$. $\square$

Section6.1 Exercise9:
First, since $det(A - xI) = det((A - xI)^T) = det(A^T - xI)$, which means $det(A - xI)$ and $det(A^T - xI)$ have the same roots. In other words, A, A^T have the same eigenvalues.

However, it is possible that A, A^T have different eigenvectors. We will construct such an example. We use example 3 in page 303. We only need to find the eigenspaces for A^T:
For $\lambda = 4$,

$$A^T - 4I = \begin{pmatrix} 3-4 & 3 \\ 2 & -2-4 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix}.$$

It is easy to see that the eigenspace is $span\{(3, 1)^T\}$, which is different from the eigenspace for A and $\lambda = 4$ given in page 303. $\square$

Section6.3 Exercise1:
a) $det(A - \lambda I) = det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0$, thus eigenvalues are ± 1.
For $\lambda_1 = 1$, we want to find the eigenspace:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}.$$

Thus we know the eigenspace is $V_1 = \{(x, y)^T | x + y = 0\} = span\{(1, -1)^T\}$, so we can pick eigenvector $\mathbf{x_1} = (1, -1)^T$. Similar computation shows that for $\lambda_2 = -1$, we can pick eigenvector $\mathbf{x_2} = (-1, 1)^T$. Thus we can set $X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Computation shows $X^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$. And we have $A = XDX^{-1}$.

d) Since A is upper-triangular, the eigenvalues are exactly the diagonal entries: 2, 1 and -1.

$\boxed{\lambda_1 = 2}$

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}.$$

From the third row we have $-3z = 0$, thus $z = 0$. Looking back at the second row we have $-y + 2z = 0$, hence $y = 0$ also. Looking back at the first row we have a null condition. Hence the eigenspace is $V_1 = \{(x, 0, 0)^T\}$. We pick a eigenvector $\mathbf{x_1} = (1, 0, 0)^T$.

$\boxed{\lambda_2 = 1}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}.$$

From the second and third row we know $z = 0$. From the first row we know the eigenspace is $V_2 = \{(x, y, 0)^T | x + 2y = 0\} = span\{(2, -1, 0)^T\}$ and pick an eigenvector $\mathbf{x_2} = (2, -1, 0)^T$.

$\boxed{\lambda_3 = -1}$

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}.$$

From the second row we know $2y + 2z = 0$ thus $y = -z$. Substitute into the first row we have $0 = 3x + 2y + z = 3x - z$. Hence $3x = z = -y$ and the eigenspace is $V_3 = \{(x, -3x, 3x)^T\} = span\{(1, -3, 3)^T\}$. Finally we set

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix}.$$

And computation gives us

$$X^{-1} = \begin{pmatrix} 1 & 2 & 5/3 \\ 0 & -1 & -1 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

You can check that $A = XDX^{-1}$. $\square$ ☞Depending on your choice of eigenvectors, the matrices X and X^{-1} may vary.

Section6.3 Exercise30:

Since A is diagonalizable, say $A = XDX^{-1}$ for some diagonal matrix D and invertible matrix X. Thus $e^A = Xe^D X^{-1}$ and to prove e^A is invertible we need only to prove e^D is invertible. From page 337 in textbook, we know $det(e^D) = e^{\lambda_1} \cdot e^{\lambda_2} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots \lambda_n}$ which is never zero, thus e^D is invertible. $\square$

Section6.4 Exercise7:

a) $(A^H)^H = \overline{(\overline{A}^T)}^T = (\overline{\overline{A}}^T)^T = A.$

b) $(\alpha A+\beta B)^H = (\overline{\alpha A+\beta B})^T = (\overline{\alpha}\overline{A}+\overline{\beta}\overline{B})^T = \overline{\alpha}\overline{A}^T + \overline{\beta}\overline{B}^T = \overline{\alpha}A^H + \overline{\beta}B^H.$
c) $(AB)^H = (\overline{AB})^T = (\bar{A}\bar{B})^T = \bar{B}^T\bar{A}^T = B^H A^H.$ $\square$