

Section 6.4 Exercise 13:

Given $U = I - 2uu^H$, by exercise 7 we have $U^H = (I - 2uu^H)^H = I^H - \bar{2}u^H u^H = I - 2uu^H = U$. This shows U is Hermitian.

$U^H U = (I - 2uu^H)(I - 2uu^H) = I - 2uu^H - 2uu^H + 4(uu^H)(uu^H)$. Now u is a unit vector, in other words $u^H u = 1$, hence $U^H U = I - 4uu^H + 4u(u^H u)u^H = I - 4uu^H + 4uu^H = I$. This shows U is unitary.

Finally, from $U^H U = I$ we know $U^{-1} = U^H = U$. That is U is its own inverse. \square

Section 6.5 Exercise 1:

Let $A = U\Sigma V^T$ be a singular value decomposition. Observe that Σ^T is a matrix whose off-diagonal entries are all 0's and whose diagonal entries are exactly the diagonal entries of Σ . In particular, diagonal entries are decreasing and positive. Thus $A^T = V\Sigma^T U^T$ is a singular value decomposition of A^T . Hence by the uniqueness of singular values, we know the singular values of A^T are the diagonal entries of $\Sigma^T = \Sigma$, which are also singular values of A . \square

Section 6.5 Exercise 2:

a) Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, then $A^T A = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$ which has eigenvalues $\lambda_1 = 10, \lambda_2 = 0$. Hence

we get the singular values of A $\sigma_1 = \sqrt{10}, \sigma_2 = 0$. (remember: we must pick $\lambda_1 \geq \lambda_2$)

We now compute the matrix V . For this we first need to find out the eigenspaces of $A^T A$.

For $\lambda_1 = 10$, the eigenspace is clearly $V_1 = \text{span}\{(1, 1)^T\}$. For $\lambda_2 = 0$, the eigenspace is

$V_2 = \text{span}\{(1, -1)^T\}$. Normalize these two eigenvectors, we get $V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Turning onto get U . First from observation 4, we have

$$\mathbf{u}_1 = A\mathbf{v}_1/\sigma_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}.$$

Now since the second singular value $\sigma_2 = 0$, observation 4 tell us no info about \mathbf{u}_2 , just like the example 1 case. We need other observations. Since A has rank 1, by observation 5 \mathbf{u}_2 is an orthonormal basis for $N(A^T) = \{(x, y)^T | x + 2y = 0\} = \text{span}\{(2, -1)^T\}$. Normalizing this vector, we have

$$\mathbf{u}_2 = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}; \text{ and set } U = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}.$$

The singular value decomposition is

$$A = U \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{pmatrix} V^T. \square$$

c) Let

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ then we have } A^T A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$$

Eigenvalues for $A^T A$ are $\lambda_1 = 16, \lambda_2 = 4$, and singular values of A are $\sigma_1 = 4, \sigma_2 = 2$. For $\lambda_1 = 16$, the eigenspace is $V_1 = \{(x, y)^T \mid -6x + 6y = 0\} = \text{span}\{(1, 1)^T\}$. And for $\lambda_2 = 4$, the eigenspace is $V_2 = \{(x, y)^T \mid 6x + 6y = 0\} = \text{span}\{(1, -1)^T\}$. Normalizing these two eigenvectors, we set

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since both the singular values are all nonzero, we can use full strength of observation 4:

$$\mathbf{u}_1 = \frac{1}{4} A \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{2} A \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Again since A has rank 2, we can use observation 5 to get the rest two column vectors of U . The null space $N(A^T) = \{(x, y, z, w)^T \mid x + 3y = 0 = 3x + y\} = \{(0, 0, z, w)^T\}$, which has a natural orthonormal basis: $\mathbf{u}_3 = (0, 0, 1, 0)^T, \mathbf{u}_4 = (0, 0, 0, 1)^T$. Thus we can set

$$U = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The singular value decomposition is

$$A = U \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} V^T. \square$$

Section 6.5 Exercise 8:

Let $A = U\Sigma V^T$ be a singular value decomposition. Notice that since A is a square $n \times n$ matrix, U, Σ and V are all $n \times n$ matrices. Moreover, since Σ is a diagonal square matrix we have $\Sigma^T = \Sigma$. Computations give $A^T = V\Sigma^T U^T = V\Sigma U^T$, $AA^T = (U\Sigma V^T)(V\Sigma U^T) = U\Sigma^2 U^T$ and $A^T A = V\Sigma^2 V^T$ since $UU^T = VV^T = I$. (because A is a square matrix, UU^T and VV^T give identity matrices of the rank) Define $X = VU^{-1} = VU^T$. Since both U and V are invertible, X is invertible and $X^{-1} = (VU^{-1})^{-1} = UV^{-1} = UV^T$. Finally $XAA^T X^{-1} = (VU^T)(U\Sigma^2 U^T)UV^T = VUU^T \Sigma^2 U^T UV^T = V\Sigma^2 V^T = A^T A$, this shows $A^T A$ and AA^T are similar. \square

Section 6.6 Exercise 1:

$$\text{a) } \begin{pmatrix} 3 & -5/2 \\ -5/2 & 1 \end{pmatrix}; \text{ b) } \begin{pmatrix} 2 & 1/2 & -1 \\ 1/2 & 3 & 3/2 \\ -1 & 3/2 & 1 \end{pmatrix}; \text{ c) } \begin{pmatrix} 1 & 1/2 & -1 \\ 1/2 & 2 & 3/2 \\ -1 & 3/2 & 1 \end{pmatrix} \square$$

Section 6.6 Exercise 4:

Since λ_1 and λ_2 are eigenvalues, a suitable change of coordinates makes the original equation $ax^2 + 2bxy + cy^2 = 1$ into this new equation $\lambda_1 x^2 + \lambda_2 y^2 = f$ for some constant f . We know $\lambda_1 \lambda_2 < 0$, which means they differ in sign. Thus this equation represents a hyperbola. \square

Section 6.6 Exercise 7:

a) Let $f(x, y) = 3x^2 - xy + y^2$ and $P = (0, 0)$. Then $f_x(x, y) = 6x - y$ and $f_y = -x + 2y$. Because $f_x(P) = f_y(P) = 0$ we know P is a stationary point. Continuing computations

give $f_{xx} = 6, f_{xy} = f_{yx} = -1, f_{yy} = 2$. Thus the Hessian of f at P is $H = \begin{pmatrix} 6 & -1 \\ -1 & 2 \end{pmatrix}$,

which has the characteristic polynomial $f(x) = x^2 - 8x + 11 = (x - (4 + \sqrt{5}))(x - (4 - \sqrt{5}))$. Since $(4 \pm \sqrt{5}) > 0$ we know H is positive definite and $f(P)$ is a local minimum.

e) Let $f(x, y, z) = x^3 + xyz + y^2 - 3x$ and $P = (1, 0, 0)$. We have $f_x = 3x^2 + yz - 3, f_y = xz + 2y, f_z = xy$ and $f_x(P) = f_y(P) = f_z(P) = 0$, hence P is a stationary point.

Also we have $f_{xx} = 6x, f_{xy} = f_{yx} = z, f_{xz} = f_{zx} = y, f_{yy} = 2, f_{yz} = f_{zy} = x, f_{zz} = 0$,

thus the Hessian of f at P is $H = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ which has the characteristic polynomial

$(6 - x)(x^2 - 2x - 1)$ and eigenvalues $6, 1 \pm \sqrt{2}$. Since eigenvalues differ in sign, P is a saddle point. \square