# Final Exam: MATH 425
## Due 11:59pm, May 19, 2009

Please do **all four** problems and email your solutions to `clein@math.uiuc.edu` attached as a pdf file (scans are perfectly acceptable) or bring them to my office May 19 at 12:00 noon. Do not consult with any sources other than your notes and your textbook (you can freely assume any result from class, homework, midterm, or text as well as standard results from calculus). You should **not** discuss these problems with anyone else.

**Problem 1**
Prove that the subset of $\mathbb{R}^3$ defined by

$$xyz = 0$$

is not a $k$–manifold for any $k$.

**Problem 2**
Let $M_{n\times n}(\mathbb{R})$ denote the $n \times n$ matrices with real coefficients, which we can (as usual) identify with $\mathbb{R}^{n^2}$. Let

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det(A) = 1\}.$$

Prove that $\mathrm{SL}_n(\mathbb{R})$ is an $(n^2 - 1)$–manifold. (Hint: You might look at your midterm.)

**Problem 3**
Consider the following differential forms on $\mathbb{R}^4$

$$\begin{aligned} \omega &= x_2x_3\,dx_1 \wedge dx_3 + 4x_1\,dx_2 \wedge dx_4 - 2x_3\,dx_3 \wedge dx_4, \\ \eta &= x_1^2\,dx_1 + 2x_3x_4\,dx_3, \\ \mu &= x_4^2\,dx_1 \wedge dx_2 \wedge dx_3 + x_3\,dx_2 \wedge dx_3 \wedge dx_4 \end{aligned}$$

(a) Compute $d\omega$, $d\eta$ and $d\mu$.

(b) Compute $\omega \wedge \eta$ and $\eta \wedge \mu$.

(c) Let $F : \mathbb{R}^2 \to \mathbb{R}^4$ be the map defined by

$$F(y_1, y_2) = (y_1^2y_2, y_1y_2, y_1 - y_2, y_1 + y_2).$$

Compute $F^*\omega$, $F^*\eta$, and $F^*\mu$.

The solutions to parts (a) and (b) should be expressed as sums of smooth functions times the *standard differential forms*

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

where $i_1 < \cdots < i_k$, and similarly for part (c) in terms of $dy_{i_1} \wedge \cdots \wedge dy_{i_k}$ with $i_1 < \cdots < i_k$.

**Problem 4**
Let $\mathbb{T}$ denote the 2–manifold in $\mathbb{R}^3$ obtained by revolving the circle in the $xz$–plane defined by the equation

$$(x-2)^2 + z^2 = 1$$

about the $z$–axis. Orient $\mathbb{T}$ so that at the point $\mathbf{p} = (3, 0, 0) \in \mathbb{T}$, the unit normal vector

$$(\,\mathbf{p}\,;\ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,)$$

is positive (see p.284 of the text). Let $\omega$ be the 2–form defined by

$$\omega = (3x^2 \cos(y) + e^{xy})\, dx \wedge dy + 17x^3\, dx \wedge dz + (x + yz^5 + \sin(z))\, dy \wedge dz.$$

Compute

$$\int_{\mathbb{T}} \omega.$$