Math 425: Midterm

Solve all problems, explain your work and write neatly.

1. (20 points) Consider the following open subsets of $\mathbb{R}^2$

$$A = \{\, (x_1, x_2) \,|\, 0 < x_2 < \pi \,\} \quad \text{and} \quad B = \{\, (y_1, y_2) \,|\, y_2 > 0 \,\}.$$

   (a) Prove that $h : A \to B$ given by

$$h(x_1, x_2) = (e^{x_1} \cos(x_2), e^{x_1} \sin(x_2))$$

   is a C^∞ diffeomorphism (the coordinate functions are obviously C^∞, you do not need to prove this).

   (b) Let

$$C = \{\, (y_1, y_2) \in B \,|\, 1 < y_1^2 + y_2^2 < e^2 \,\}$$

   Compute

$$\int_C \frac{1}{(y_1^2 + y_2^2)^{3/2}}.$$

2. (15 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogenous C^1 function of degree $d \geq 1$. That is, f is C^1 and

$$f(t\mathbf{x}) = t^d f(\mathbf{x})$$

   for all $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Prove that if $f(\mathbf{x}) \neq 0$, then $Df(\mathbf{x}) \neq 0$.

   Hint: For any $\mathbf{x} \in \mathbb{R}^n$ with $f(\mathbf{x}) \neq 0$, consider the C^1 function $g : \mathbb{R} \to \mathbb{R}^n$ given by $g(t) = t\mathbf{x}$.

3. (5 points) If $C \subset \mathbb{R}^n$ is a compact set with empty interior which does *not* have measure zero, prove that C is not rectifiable. For one bonus point, construct such a set.