

Differentiable Manifolds: Problem set 11

Due Monday November 17

Read GP §4.1, Warner 34–50. Do problems:
GP. §3.5: 18, 19.

1. Compute the Euler characteristic of the complex projective space $\mathbb{C}P^n$ (hint: consider the map $[z_1 : \dots : z_n] \mapsto [z_1 : 2z_2 : \dots : nz_n]$).
2. Draw a smooth vector field in \mathbb{R}^2 with isolated zeros of index (a) -2; (b) +3; (c) 0; What do the flow lines look like?
3. Use flows to prove that a connected manifold is *homogeneous*, i.e. for all $p, q \in M$, there exists a diffeomorphism $f : M \rightarrow M$ with $f(p) = q$.
4. Prove that any vector field on a compact manifold without boundary is complete (which then proves Theorem II.37 from class).
5. (This problem will not be graded) Given $A \in M_{n \times n}(\mathbb{R})$, define

$$\exp(A) = e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

- a. Check that

$$\exp : M_{n \times n}(\mathbb{R}) \rightarrow M_{n \times n}(\mathbb{R})$$

is well defined, and in fact smooth. Hint: consider coordinates $\{y_{ij}\}_{i,j=1,\dots,n}$ on $M_{n \times n}(\mathbb{R})$, and verify that the coordinate functions of \exp are convergent power series.

- b. Check that if A and B commute (i.e. $AB = BA$), then

$$\exp(A + B) = \exp(A) \exp(B)$$

- c. Verify that \exp maps $M_{n \times n}(\mathbb{R})$ to $GL_n(\mathbb{R})$. Moreover, viewing a 1-dimensional subspace of $M_{n \times n}(\mathbb{R})$ as an abelian group, check that the restriction of \exp to such a group defines a homomorphism to $GL_n(\mathbb{R})$.

- d. Recall from homework 5 the construction of the vector field ξ^A from $A \in M_{n \times n}(\mathbb{R})$. Verify that the flow defined by ξ^A

$$\phi^A : \mathbb{R} \times GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$$

is given by

$$\phi_t^A(X) = X \exp(tA)$$

- e. Observe (nothing to do here) that the action of \mathbb{R} defined by ϕ^A is actually the restriction to the subgroup of part (c.) of the right action of $GL_n(\mathbb{R})$ on itself.