Differentiable Manifolds: Problem set 12

Due Wednesday, December 10

Read Warner Chapter 2, page 55–72, GP. 4.1–4.4 (NOTE: we are following Warner's conventions).

1. Let V be a finite dimensional vector space over $\mathbb{R}$. Prove that $\eta_1, ..., \eta_p \in V^*$ are linearly independent if and only if

$$\eta_1 \wedge \cdots \wedge \eta_p \in \Lambda_p(V^*)$$

   is nonzero.

2. Let V be an n–dimensional vector space over $\mathbb{R}$. Prove the Determinant Theorem: Suppose $\omega \in \Lambda_n(V^*)$ and $\Phi : V \to V$ is a linear map. Then

$$\Phi^*(\omega) = \det(\Phi)\omega.$$

3. Let V be an n–dimensional vector space over $\mathbb{R}$.

   (a.) Let $\omega \in \Lambda_n(V^*)$ with $\omega \neq 0$. Prove that the ordered bases $v_1, ..., v_n$ for V is equivalent to the ordered basis $u_1, ..., u_n$ if and only if $\omega(v_1, ..., v_n)$ and $\omega(u_1, ..., u_n)$ have the same sign.

   (b.) Prove that an orientation on V determines one on $\Lambda_n(V^*)$, and conversely, by declaring $\omega \in \Lambda_n(V^*)$, $\omega \neq 0$, to be a positively oriented basis for $\Lambda_n(V^*)$ if and only if $\omega(v_1, ..., v_n) > 0$ for a positively oriented ordered basis $v_1, ..., v_n$ for V.

4. Prove that an n–manifold M is orientable if and only if there exists a nowhere vanishing n–form $\omega \in \Omega^n(M)$ (Hints: (1) look at the previous problem; (2) consider a partition of unity.)

5. Assuming M is orientable, define an equivalence relation on nowhere vanishing n–forms declaring ω to be equivalent to η if and only if $\omega = f\eta$ for some $f \in C^\infty(M)$ with $f > 0$. Explain how an orientation on M is equivalent to a choice of equivalence class of nowhere vanishing n–form. A particular choice of nowhere vanishing n–form is called a *volume form*.

6. Prove that if $f : M \to N$ and $g : N \to L$ are smooth maps, and $\eta \in \Omega^*(L)$, then

$$(g \circ f)^*(\eta) = f^*(g^*(\eta)).$$

7. Suppose $f : M \to N$ is a smooth map. Suppose we have local coordinates $(x_1, ..., x_n) : U \to \mathbb{R}^n$ on M and $(y_1, ..., y_k) : V \to \mathbb{R}^k$ on N with $f(U) \subset V$. For $\eta \in \Omega^p(N)$, we can write $\eta|_V$ as

$$\eta|_V = \sum_{i_1 < \cdots < i_p} \eta_{i_1 \cdots i_p} dy_{i_1} \wedge \cdots \wedge dy_{i_p}.$$

(a.) Writing $f_i = f^*(y_i) = y_i \circ f$, prove

$$f^*(\eta|_V) = f^* \left(\sum_{i_1 < \cdots i_p} \eta_{i_1 \cdots i_p} dy_{i_1} \wedge \cdots \wedge dy_{i_p} \right)$$

$$= \sum_{i_1 < \cdots < i_p} \sum_{j_1, ..., j_p} (\eta_{i_1 \cdots i_p} \circ f) \frac{\partial f_{i_1}}{\partial x_{j_1}} \cdots \frac{\partial f_{i_p}}{\partial x_{j_p}} dx_{j_1} \wedge \cdots \wedge d_{j_p}.$$

Note that the right-hand side is a sum over two types of indices: The first is a sum over increasing indices $1 \leq i_1 < \cdots < i_p \leq n$, while the second is a sum over *all* indices $1 \leq j_1, ..., j_p \leq n$. Thus, we have not expressed $f^*(\eta|_V)$ as a linear combination of the basis p–forms $\{dx_{j_1} \wedge \cdots \wedge dx_{j_p}\}_{j_1 < \cdots < j_p}$ for $\Omega^p(U)$, unless $p = 1$.

(b.) This is overly complicated for $p = 1$: write this more succinctly in this case (hint: you did this a while ago).

(c.) For $p = 2$, write the right hand side in terms of $\{dx_{j_1} \wedge dx_{j_2}\}_{j_1 < j_2}$.

8. Consider the standard coordinates x, y, z on $\mathbb{R}^3$. Let

$$\begin{aligned} \eta &= x^2 dx + z^2 dy + y^2 dz \\ \mu &= (x - y) dx \wedge dy + (x + z^2) dy \wedge dz \\ \xi &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \end{aligned}$$

Compute each of the following

(a.) $\eta \wedge \mu$

(b.) $d\eta$

(c.) $d\mu$

(d.) $L_\xi \mu$

For part (d.), compute this in two ways: (1) using the definition of Lie derivative and (2) using the Cartan formula, $L_\xi = d\iota_\xi + \iota_\xi d$.

9. Consider the $(n-1)$–form on $\mathbb{R}^n$

$$\omega = \sum_{i=1}^{n} (-1)^{i+1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

(where as usual $\widehat{\cdot}$ means omit $\cdot$). Let $f : S^{n-1} \to \mathbb{R}^n$ be the inclusion. Prove that $f^*(\omega)$ is a nowhere vanishing $(n-1)$-form.