Differentiable Manifolds: Problem set 2

Due Monday, September 15

Read GP §1.1 and §1.2 again. Do problems 4, 8, 9, 10, and 11 from §1.2 (use our definition of tangent vector and tangent space).
Read Warner 1–19 (so, read 1–11 again paying particular attention to the discussion of partition of unity).

1. If $M \subset \mathbb{R}^N$ is a smooth manifold in the sense of GP, then it is also a smooth manifold in our sense as you proved on the last homework. In GP, the tangent space at $m \in M$ is defined to be a linear subspace of $\mathbb{R}^N$. Prove that the map $[\gamma] \to \gamma'(0) \in \mathbb{R}^N$ defines an isomorphism from the tangent space $T_m M$ in our sense to the tangent space in the sense of GP. Again, it might be convenient to define something like $V_m M \subset \mathbb{R}^N$ to be the GP tangent space to $M$ at $m$, leaving $T_m M$ to denote our tangent space. After this problem has been completed, this notation can be safely abandoned.

2. Suppose that $f : M \to N$ is a smooth map and

$$\phi = (x_1, ..., x_n) : U \to \mathbb{R}^n \text{ and } \psi = (y_1, ..., y_k) : V \to \mathbb{R}^k$$

are local coordinates about $p$ in $M$ and $f(p)$ in $N$ respectively with $f(U) \subset V$. We can write

$$\psi \circ f = (f_1(x_1, ..., x_n), ..., f_k(x_1, ..., x_n)).$$

where $f_j = y_j \circ f \circ \phi^{-1}$ is a smooth function on an open set in euclidean space, and $f_j(x_1, ..., x_n) = f_j \circ \phi$ is a *smooth function on* $U$.

In this exercise, you will verify that with respect to the standard bases given by the local coordinates, the derivative looks just like the usual jacobian matrix.

(a.) Prove that

$$df_p : T_p M \to T_{f(p)} N$$

is given

$$df_p \left( \left. \frac{\partial}{\partial x_j} \right|_p \right) = \sum_i \left. \frac{\partial f_i}{\partial x_j} \right|_p \left. \frac{\partial}{\partial y_i} \right|_{f(p)}.$$

(b.) Prove that

$$(df_p)^* : T^*_{f(p)} N \to T^*_p M$$

is given by

$$(df_p)^* \left( dy_j|_{f(p)} \right) = \sum_i \left. \frac{\partial f_j}{\partial x_i} \right|_p dx_i|_p$$

3. The set of complex lines in $\mathbb{C}^{n+1}$ is called **complex projective space** and is denoted $\mathbb{CP}^n$. It is the set of equivalence classes

$$\{[z_0 : .... : z_n] \,|\, (z_0, ..., z_n) \in \mathbb{C}^{n+1} \setminus \{0\}\}$$

where $[z_0 : ... : z_n] = [w_0 : .... : w_n]$ if and only if there exists $t \in \mathbb{C}^*$ so that $z_j = tw_j$ for each $j = 0, ..., n$.
Identifying $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ by

$$(z_0, ..., z_n) = (x_0 + iy_0, ..., x_n + iy_n) \mapsto (x_0, y_0, ..., x_n, y_n)$$

prove that $\mathbb{CP}^n$ can be made into a smooth (real) $2n$–manifold so that the projection

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$$

is a smooth map.

Hint: Define the same kinds of charts $\psi_j([z_0 : ... : z_n]) = (z_0, ..., \hat{z}_j, ..., z_n)$ as for real projective space.

You do not need to hand in any of the following problems. However, I strongly recommend you work through them as they provide a wealth of examples of smooth manifolds.

4. Suppose $G$ acts properly discontinuously and freely on a locally compact Hausdorff topological space $X$.

   (a.) Prove that $X/G$ is Hausdorff.

   (b.) Prove that the quotient map

   $$\pi : X \to X/G$$

   is a **covering map**: for every $x \in X/G$, there is a neighborhood $U$ of $x$ so that the preimage of $U$ is a disjoint union of open sets

   $$\pi^{-1}(U) = \bigsqcup_{\alpha \in J} U_\alpha$$

   with the property that $\pi|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism for every $\alpha \in J$.

   (c.) Prove that if $X$ has a countable basis for the topology, then so does $X/G$.

   Hints: Recall that a locally compact Hausdorff space has the property that in any neighborhood $U$ about a point $x$, there is smaller neighborhood $V$ of $x$ with $\overline{V} \subset U$ and $\overline{V}$ compact.

5. Suppose that $M$ is a smooth manifold with a properly discontinuous free smooth action of $G$ on $M$. Define a smooth structure on $M/G$ for which

$$\pi : M \to M/G$$

is smooth.

6. Suppose that $\pi : M \to N$ is a covering map of topological spaces (see above), and that $N$ is a smooth manifold. If $M$ has countable fibers (meaning $\pi^{-1}(p)$ is countable for every $p \in N$) then construct a smooth structure on $M$ making it into a smooth manifold for which $\pi$ is a smooth map.

7. Verify that the following actions are smooth, properly discontinuous, and free (so the quotients are smooth manifolds).

   (a.) $\mathbb{Z}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $A \cdot B = A + B$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is the **$n$-torus**, and is denoted $\mathbb{T}^n$. Check that this is diffeomorphic to the $n$-fold product of $S^1$ with itself. This is an example of a **homogeneous space**.

   (b.) Identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ as above we can view $S^{2n+1} \subset \mathbb{C}^n$ as

   $$S^{2n+1} = \{(z_1, ..., z_n) \mid \sum |z_j|^2 = 1\}$$

   Let $m$ be a positive integer and $\ell_1, ..., \ell_n$ be positive integers relatively prime to $m$. Define an action of the cyclic group $\mathbb{Z}/m$ (the additive group of integers modulo $m$) on $S^{2n+1}$ by

   $$[k] \cdot (z_1, ..., z_n) = \left(e^{\frac{2k\pi i \ell_1}{m}} z_1, ..., e^{\frac{2k\pi i \ell_n}{m}} z_n\right)$$

   for $[k] \in \mathbb{Z}/m$ (the residue of $k \in \mathbb{Z}$ modulo $m$). The quotient space is denoted $L_m(\ell_1, ..., \ell_n)$ and is called a **lens space**.
   Hint: It might be useful in this exercise to think of $S^{2n+1}$ as a smooth manifold in the sense of GP.

   (c.) $GL_n(\mathbb{Z}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ by matrix multiplication $A \cdot B = AB$. This quotient is another example of a homogeneous space.