Differentiable Manifolds: Problem set 5

Due Monday October 6

Read GP §2.1–2.2. Do problems:
§2.1: 4, 5, 9;

1. Submanifolds $N, P \subset M$ are **transverse**, written $N \pitchfork P$ if for every $m \in N \cap P$
$$T_m N + T_m P = T_m M.$$
Equivalently, the embedding of P into M is transverse to N (or equivalently, the embedding of N into M is transverse to P). Here embedding means injective immersion for which the inclusion is a homeomorphism onto its image.

   Prove the following local transversality theorem:

   Theorem *If $N^k, P^r \subset M^n$ are embedded submanifolds and $N \pitchfork P$, then for every $m \in N \cap P$, there is a coordinate chart*
$$\phi : U \to (-1,1)^n$$
about m in M, so that
$$\phi(N \cap U) = \{(a_1, ..., a_k, 0, ..., 0) \,|\, a_j \in (-1,1)\}$$
$$\phi(P \cap U) = \{(0, ..., 0, a_{n-r+1}, ..., a_n) \,|\, a_j \in (-1,1)\}$$

2. Let $U \subset \mathbb{R}^3$ and
$$\xi = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} - zx \frac{\partial}{\partial z} \text{ and } \eta = y^2 \frac{\partial}{\partial x} - xyz \frac{\partial}{\partial z}$$
compute $[\xi, \eta]$.

3. Let M be a smooth manifold and $\xi, \eta, \zeta \in \mathfrak{X}(M)$, smooth vector fields on M. Verify the **Jacobi identity**:
$$[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0$$
Along with the property $[\xi, \eta] = -[\eta, \xi]$, this shows that the (real) vector space $\mathfrak{X}(M)$ with the bilinear operation $[\cdot, \cdot]$ is a **Lie algebra**.

4. Recall that we have a canonical identification $T_g(\mathrm{GL}_n(\mathbb{R})) = \mathrm{M}_{n\times n}(\mathbb{R})$ for any $g \in \mathrm{GL}_n(\mathbb{R})$. Given $A \in \mathrm{M}_{n\times n}(\mathbb{R})$, we can construct a smooth vector field on $\mathrm{GL}_n(\mathbb{R})$

$$\xi^A : \mathrm{GL}_n(\mathbb{R}) \to T\mathrm{GL}_n(\mathbb{R}) = \mathrm{GL}_n(\mathbb{R}) \times \mathrm{M}_{n\times n}(\mathbb{R})$$

by

$$\xi^A(g) = (g, gA)$$

We have global coordinates x_{ij} on $\mathrm{M}_{n\times n}(\mathbb{R})$, and hence also on $\mathrm{GL}_n(\mathbb{R})$, with $x_{ij}(g) = g_{ij}$. We can thus express ξ^A in terms of the associated basis of vector fields $\frac{\partial}{\partial x_{ij}}$ as

$$\xi^A = \sum_{i,j,k} x_{ik} A_{kj} \frac{\partial}{\partial x_{ij}}.$$

Therefore, ξ^A is a smooth vector field on $\mathrm{GL}_n(\mathbb{R})$.

Note that left multiplication by g

$$\ell_g : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$$

$$\ell_g(h) = gh$$

is actually the restriction to $\mathrm{GL}_n(\mathbb{R})$ of a linear map $\mathrm{M}_{n\times n}(\mathbb{R}) \to \mathrm{M}_{n\times n}(\mathbb{R})$. So the derivative map

$$d(\ell_g)_h : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathrm{M}_{n\times n}(\mathbb{R})$$

is just given by

$$d(\ell_g)_h(B) = gB$$

for any $h \in \mathrm{GL}_n(\mathbb{R})$.

a. Check that

$$\beta : \mathrm{M}_{n\times n}(\mathbb{R}) \to \mathfrak{X}(\mathrm{GL}_n(\mathbb{R}))$$

given by

$$\beta(A) = \xi^A$$

is a *linear* injection. Thus the image of β is a finite dimensional subspace of $\mathfrak{X}(\mathrm{GL}_n(\mathbb{R}))$, which is denoted $\mathfrak{gl}_n(\mathbb{R})$.

b. Check that for every $g \in \mathrm{GL}_n(\mathbb{R})$ the diffeomorphism ℓ_g preserves ξ^A (that is, ξ^A is ℓ_g–related to itself).

c. Check that $\mathfrak{gl}_n(\mathbb{R})$ is closed under the Lie bracket, and in fact:

$$[\xi^A, \xi^B] = \xi^{AB-BA}$$

(it is a big computation.. don't be scared). This last matrix $AB-BA$ is often denoted $[A,B] = AB - BA$, so that the equation becomes

$$[\xi^A, \xi^B] = \xi^{[A,B]}$$

Thus, $\mathfrak{gl}_n(\mathbb{R})$ is a finite dimensional Lie subalgebra of $\mathfrak{X}(\mathrm{GL}_n(\mathbb{R}))$ and is called **the Lie algebra** of $\mathrm{GL}_n(\mathbb{R})$.

d. Check that if A and B are in $T_I(\mathrm{SL}_n(\mathbb{R}))$, i.e. they have trace zero, then $[A, B] \in T_I(\mathrm{SL}_n(\mathbb{R}))$. In particular, after restricting both domain and range, β describes a linear injection of $T_I(\mathrm{SL}_n(\mathbb{R}))$ into $\mathfrak{X}(\mathrm{SL}_n(\mathbb{R}))$ and the image is also closed under Lie bracket. This image $\mathfrak{sl}_n(\mathbb{R})$ is the Lie algebra of $\mathrm{SL}_n(\mathbb{R})$.

e. Similarly, for the orthogonal group $\mathrm{O}_n(\mathbb{R})$, check that if A and B are in $T_I(\mathrm{O}_n(\mathbb{R}))$, i.e. $A + A^t = 0$ and $B + B^t = 0$ (see GP problem 10, §1.5), then $[A, B] \in T_I(\mathrm{O}_n(\mathbb{R}))$. Again, after restricting appropriately, β defines a linear injection of $T_I(\mathrm{O}_n(\mathbb{R}))$ into $\mathfrak{X}(\mathrm{O}_n(\mathbb{R}))$. The image $\mathfrak{o}_n(\mathbb{R})$ is the Lie algebra of $\mathrm{O}_n(\mathbb{R})$.

Note that the Lie algebras $\mathfrak{gl}_n(\mathbb{R})$ is isomorphic to $M_{n\times n}(\mathbb{R})$ with the bracket given by $[A, B] = AB - BA$. Similarly, $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{o}_n(\mathbb{R})$ are isomorphic to Lie subalgebras of $M_{n\times n}(\mathbb{R})$.