Differentiable Manifolds: Problem set 9

Due Monday November 3

Read GP §3.2-3.4. Do problems:
§3.2: 8, 14, 16, 27. §3.3: 1, 6, 10, 14, 15.

Problems 2 and 3 below do not need to be turned in.

1. Suppose $f : M \to N$ is a *covering map*: for every point $p \in N$ there is a neighborhood U of p so that

$$f^{-1}(U) = \bigsqcup_{\alpha \in J} V_\alpha$$

with V_α open in M and $f|_{V_\alpha} : V_\alpha \to U$ a diffeomorphism, for all $\alpha \in J$ (the index set J is at most countably infinite so that M is in fact a manifold).

If N is orientable, prove that M is also orientable. If M and N are both compact and N is connected, prove that M can be oriented so that $f^{-1}(p)$ consists of exactly $deg(f)$ points. (So the number of sheets of the covering is precisely the degree.)

Remark: As we have seen with $S^n \to \mathbb{RP}^n$, with n even, the converse is not true.

2. Suppose M is a compact oriented n–manifold (without boundary) and $P_1, P_2 \subset M$ are oriented, closed k–dimensional submanifolds (also without boundary). Say that P_1 and P_2 are *cobordant* in M if there is a compact oriented manifold with boundary W for which the boundary is a disjoint union $\partial W = \partial_1 W \sqcup \partial_2 W$ and a smooth map

$$f : W \to M$$

for which the restrictions of ∂f to $\partial_1 W$ and $-\partial_2 W$ are orientation preserving diffeomorphisms onto P_1 and P_2, respectively. Here the orientations on $\partial_1 W$ and $\partial_2 W$ are the boundary orientations, and hence $-\partial_2 W$ has this orientation reversed.

Now suppose N is an oriented manifold, and $Q \subset N$ is a closed oriented submanifold (both without boundary). Further assume that

$$h_1, h_2 : M \to N$$

are homotopic maps transverse to Q and give $P_i = h_i^{-1}(Q)$, for $i = 1, 2$, the induced orientations. Prove that P_1 is cobordant to P_2 in M.

3. Suppose that S is a compact connected oriented surface without boundary and $m \in S$ is a basepoint. Assume that every homomorphism $\phi : \pi_1(S, m) \to \mathbb{Z}$ can be realized by a smooth map $f^\phi : S \to S^1$—by this we mean that after identifying $\pi_1(S^1) \cong \mathbb{Z}$, $f^\phi_* = \phi : \pi_1(S, z) \to \mathbb{Z}$ (can you prove this?). Making f transverse to a point $p \in S^1$, we get $P_\phi = (f^\phi)^{-1}(p)$, a closed oriented 1-dimensional submanifold.

   Prove that
$$\phi([\gamma]) = I(\gamma, P^\phi)$$
   where $[\gamma] \in \pi_1(S, m)$ is represented by a smooth loop $\gamma : S^1 \to S$.

   Conversely, given a closed oriented embedded 1–dimensional submanifold P, we can define
$$\phi([\gamma]) = I(\gamma, P).$$
   Prove that there exists a map $f^\phi : S \to S^1$ for which $P = (f^\phi)^{-1}(p)$.

   Finally, assuming that two maps f^ϕ and h^ϕ realizing the homomorphism ϕ are homotopic, prove that any two closed oriented embedded 1–dimensional submanifolds P_1 and P_2 defining the same homomorphism $\phi : \pi_1(S, z) \to \mathbb{Z}$ are cobordant in S.

   Remark: This is an instance of Poincaré duality. Namely, a 1–dimensional cohomology class, which can be thought of as a homomorphism ϕ from $\pi_1(S, m)$ to $\mathbb{Z}$, is dual via intersection theory to a 1-dimensional homology class P_ϕ, and conversely.