Differential Geometry: Problem set 6

November 17, 2006

Due Friday, November 13

1. Let V be a finite dimensional vector space. Verify that $v_1 \wedge ... \wedge v_k \in \Lambda_k V$ is nonzero if and only if $v_1, ..., v_k$ are linearly independent.

2. In class we defined a canonical isomorphism indirectly

$$\Lambda^k V \cong Alt_k(V; \mathbb{R})$$

where $Alt_k(V; \mathbb{R})$ was the space of alternating maps from the k-fold product of V with itself to $\mathbb{R}$. However, the isomorphism was somewhat indirect. To check your understanding, given

$$\omega = \sum_{i_1 < ... < i_k} \omega_{i_1,...,i_k} e^{i_1} \wedge ... \wedge e^{i_k} \in \Lambda^k V$$

write down the effect of the associated alternating function on the k–tuple $(e_{j_1}, ..., e_{j_k})$.

3. Let U, V and W be finite dimensional vector spaces and $T : U \to V$ and $S : V \to W$ are linear maps. Check that the induced maps

$$T^* : \Lambda^* V \to \Lambda^* U \quad S^* : \Lambda^* W \to \Lambda^* V \quad (ST)^* : \Lambda^* W \to \Lambda^* U$$

satisfy

$$(ST)^* = T^* S^*$$

4. Suppose M, N and L are smooth manifolds with smooth maps

$$F : M \to N \quad G : N \to L$$

check that the induced maps

$$F^* : \Omega^* N \to \Omega^* M \quad G^* : \Omega^* L \to \Omega^* N \quad (GF)^* : \Omega^* L \to \Omega^* M$$

satisfy

$$(GF)^* = F^* G^*$$

5. Verify that if $F : M \to N$ is a smooth map and $\omega \in \Omega^p(N)$, then

$$F^*(d\omega) = dF^*(\omega)$$

6. In $\mathbb{R}^n$, define an $n-1$-form

$$\omega = \sum_{i=1}^{n} (-1)^{i+1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

(where as usual $\widehat{\cdot}$ means omit $\cdot$). Consider the standard embedding

$$f : S^{n-1} \subset \mathbb{R}^n$$

and prove that $f^*(\omega)$ is a nowhere vanishing top-form, and so S^n is orientable.

7. If n is odd, show that $\mathbb{RP}^n$ is orientable. Show that when n is even it is non-orientable. In both cases, it is useful to consider the quotient map $S^n \to \mathbb{RP}^n$.

8. Let M be any smooth manifold and $\pi : T^*M \to M$ its cotangent bundle. There is a canonically defined 1-form $\Theta \in \Omega^1(T^*M)$ defined by

$$\Theta_{m,\alpha} = (d\pi_{(m,\alpha)})^* \alpha$$

That is, at the point (m, α), $\Theta_{(m,\alpha)} \in T^*_{(m,\alpha)} T^*M$ is the pull-back by $d\pi^*$ of the covector $\alpha \in T^*_m(M)$. Verify in local coordinates that Θ is indeed a (smooth) 1-form. Specifically, if $x^1, ..., x^n$ are coordinates on M and $x^1, .., x^n, y^1, ..., y^n$ are the associated coordinates on T^*M, express Θ in terms of these coordinates and verify it is smooth. Also compute (in these coordinates) $\omega = d\Theta$ and verify that the n-fold wedge product

$$\omega \wedge \dots \wedge \omega$$

is a nowhere vanishing top form.