Differential Geometry: Problem set 8

November 29, 2006

Due Friday, December 8

1. Complete the proof of Proposition 6.4:
   (1) $\frac{D}{dt}(\xi + \eta) = \frac{D\xi}{dt} + \frac{D\eta}{dt}$
   (2) $\frac{D}{dt}(\alpha\xi) = \frac{d\alpha}{dt}\xi + \alpha\frac{D\xi}{dt}$
   for any vector fields ξ, η over a curve $\gamma : (0, 1) \to M$ and $\alpha \in C^\infty((0, 1))$.

2. Compute the Christoffel symbols Γ^k_{ij} for Euclidean space $\mathbb{R}^n$, with the standard metric (given by $g_{ij} = \delta_{ij}$).

3. Compute the Christoffel symbols Γ^k_{ij} for the hyperbolic space $\mathbb{H}^n$ defined in the previous homework.

If (M, g) is a Riemannian metric, then for every $m \in M$, g_m defines a nondegenerate pairing

$$T_m M \times T_m M \to \mathbb{R}$$

and thus a canonical isomorphism $T_m M \cong T^*_m M$. This defines an isomorphism

$$\mathfrak{X}(M) \cong \Omega^1 M$$

defined by

$$\xi \mapsto g(\xi, \cdot)$$

we will refer to ξ and $g(\xi, \cdot)$ as being *dual* to each other.

4 (a) If (M^n, g) is a Riemannian manifold with local coordinates $x^1, ..., x^n$ on an open set U, prove that there exists vector fields $\xi_1, ..., \xi_n \in \mathfrak{X}(U)$ so that for every $m \in M$, $\xi_1|_m, ..., \xi_n|_m$ is an orthonormal basis for $T_m M$.

(b) Let $\theta^1, ..., \theta^n \in \Omega^1 M$ be the dual 1–forms to $\xi_1, ..., \xi_n$. Prove that $\omega = \theta^1 \wedge ... \wedge \theta^n$ is a nowhere vanishing top-form.

(c) Prove that if ω and ω' are defined as above with respect to two different coordinate charts, then $\omega = \pm\omega'$.

5 If $f \in C^\infty(M)$, then define the gradient of f to be the dual of df

$$g(\text{grad}(f), \cdot) = df$$

Let ϕ_t be the flow associated to grad(f), and prove that $f(\phi_t(m)) > f(\phi_s(m))$ if $t \geq s$ and $\text{grad}(f)|_m \neq 0$.

6 Let M be closed (compact without boundary) and oriented and $\omega \in \Omega^n M$ be the representative top-form of the orientation which agrees (up to sign) with the form defined in 4(b). If $\xi \in \mathfrak{X}(M)$, the *divergence* of ξ is defined by

$$\text{div}(\xi)\,\omega = L_\xi \omega \in C^\infty(M)$$

Recall that $L = d\iota + \iota d$, so

$$L_\xi \omega = d\iota_\xi(\omega)$$

(a) Prove that

$$df \wedge \iota_{\text{grad}(h)}\omega = g(\text{grad}(f), \text{grad}(h))$$

Hint: For each $m \in M$, you can assume grad$(h)|_m = c\,\xi_1|_m$ as in 4(a), for some $c \in \mathbb{R}$ and some coordinate chart.

(b) Define $\Delta : C^\infty(M) \to C^\infty(M)$ by

$$\Delta(f) = \text{div}(\text{grad}(f))$$

Prove

$$\Delta(fh) = f\Delta h + h\Delta f + 2g(\text{grad}(f), \text{grad}(h)).$$

A function $f \in C^\infty(M)$ is called *subharmonic* if $\Delta(f) \geq 0$; if $\Delta(f) = 0$, f is called *harmonic.*

(c) Prove that if f is a subharmonic function, then f is harmonic. *Hint: Stokes Theorem.*

(d) Prove that if f is harmonic, then f is constant. *Hint: Consider the function $f^2/2$, use Stokes again, and part (b).*