

KLEINIAN GROUPS WITH DISCRETE LENGTH SPECTRUM

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ABSTRACT. We characterize finitely generated torsion-free Kleinian groups whose real length spectrum (without multiplicities) is discrete.

Given a Kleinian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{Isom}_+(\mathbb{H}^3)$, we define the *real length spectrum of Γ without multiplicities* to be the set of all real translation lengths of elements of Γ acting on \mathbb{H}^3

$$\mathfrak{L}(\Gamma) = \{l(\gamma) \mid \gamma \in \Gamma\} \subset \mathbb{R}$$

Equivalently, $\mathfrak{L}(\Gamma) - \{0\}$ is the set of all lengths of closed geodesics in the quotient orbifold \mathbb{H}^3/Γ . Kim [8] has shown that if G is the fundamental group of a closed surface and $\Lambda \subset \mathbb{R}$ is discrete, i.e. has no accumulation points in \mathbb{R} , then there exist only finitely many Kleinian groups, up to conjugacy, which are isomorphic to G and have real length spectrum Λ .

In this paper we characterize finitely generated, torsion-free Kleinian groups whose real length spectrum is discrete. In the process, we obtain a sharper version of the Covering Theorem which may be of independent interest. As a corollary of our main result and this new Covering Theorem, we see that geometrically infinite hyperbolic 3-manifolds have discrete length spectrum if and only if they arise as covers of finite volume hyperbolic orbifolds. Our work can thus be viewed as part of a family of results, including the Covering Theorem, which exhibit the very special nature of those geometrically infinite hyperbolic 3-manifolds with finitely generated fundamental group which cover finite volume hyperbolic orbifolds.

Main Theorem. *Suppose that Γ is a finitely generated, torsion-free Kleinian group. Then the real length spectrum $\mathfrak{L}(\Gamma)$ is discrete if and only if either*

- (1) Γ is geometrically finite,
- (2) there exists a hyperbolic 3-manifold $M = \mathbb{H}^3/G$ which fibers over the circle and Γ is a fiber subgroup of G , or
- (3) there exists a hyperbolic 3-manifold $M = \mathbb{H}^3/G$ which fibers over $S^1/\langle z \mapsto \bar{z} \rangle$ and Γ is a singular fiber subgroup of G .

We recall that if a hyperbolic 3-manifold $M = \mathbb{H}^3/G$ fibers over a 1-orbifold Q , then a fiber subgroup of G is the fundamental group of the pre-image of a regular point of Q , while a singular fiber subgroup is the fundamental group of the pre-image of a singular point of Q . (Notice that one only really obtains a conjugacy class of subgroups.)

Proof. We refer the reader to [3] and [4] for details and terminology. Let $N = \mathbb{H}^3/\Gamma$, let $\epsilon > 0$ be less than the 3-dimensional Margulis constant, and let N_ϵ^0 denote

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the complement of the ϵ -cuspidal thin part of N . By the Tameness Theorem (see Agol [1] or Calegari-Gabai [2]), N is topologically tame.

It is well-known that a geometrically finite Kleinian group has discrete real length spectrum. We sketch a brief proof for the reader's convenience. If Γ is geometrically finite, then there exists a compact subset K of $N = \mathbb{H}^3/\Gamma$ such that any closed geodesic intersects K . (One may, for example, take K to be the intersection of the convex core of N with N_ϵ^0 .) If $\mathfrak{L}(\Gamma)$ were indiscrete, then there would exist an infinite collection of distinct closed geodesics of length at most L (for some $L > 0$) all of which would intersect K . If this were the case, some sequence of closed geodesics in N would have to accumulate at a closed geodesic, which is impossible. Therefore, $\mathfrak{L}(\Gamma)$ is discrete.

Since a fiber subgroup, as described in (2), or singular fiber subgroup, as described in (3), is a subgroup of a geometrically finite Kleinian group, they must also have discrete real length spectrum.

We now assume that Γ is a geometrically infinite Kleinian group with discrete real length spectrum and prove that Γ is either a fiber subgroup or a singular fiber subgroup of the fundamental group of a finite volume hyperbolic 3-manifold which fibers over a 1-orbifold. Let $U \cong \overline{S} \times [0, \infty) \subset N_\epsilon^0$ be a neighborhood of a geometrically infinite end of N_ϵ^0 and let $U_p \cong S \times [0, \infty) \subset N$ be its parabolic extension. By the main result of [3], there exists a sequence of useful simplicial hyperbolic surfaces $f_n : X_n \rightarrow U_p$, each properly homotopic in U_p to $S \times \{0\}$, which exits the end.

As the length spectrum of Γ is discrete, there is a uniform positive lower bound on the length of any closed geodesic in N . Therefore, there is a uniform positive lower bound on the length of any closed geodesic on X_n whose image under f_n is essential and not an accidental parabolic (i.e. not homotopic into a cusp of N). On the other hand, Lemma 7.1 in [3] guarantees that given any $A > 0$, there exists $n_A > 0$ so that for all $n \geq n_A$, any closed geodesic in X_n which is inessential or accidentally parabolic under f_n must have length at least A on X_n . Therefore, there is a uniform lower bound on the length of a closed geodesic on any X_n .

Lemma 7.1 in [4] then gives that there exists $C > 0$ and, for all n , a point x_n in the ϵ -thick part of X_n and a minimal generating set $\{a_1^n, \dots, a_m^n\}$ for $\pi_1(X_n, x_n)$ all of which are represented by curves (based at x_n) of length at most C ; see [4] for definitions. By passing to a subsequence, we may assume that all the minimal generating sets are topologically equivalent. Therefore, there exist markings $g_n : S \rightarrow X_n$ so that if $s \in \pi_1(S)$, then $(g_n)_*(s)$ has a representative in X_n having length bounded independent of n (but not independent of s). Composing, we obtain a sequence of representations

$$\rho_n = (f_n \circ g_n)_* : \pi_1(S) \rightarrow \Gamma \subset \mathrm{PSL}_2(\mathbb{C})$$

A result of Kim [7] gives that there exists a finite set $\{s_1, \dots, s_k\}$ of elements of $\pi_1(S)$ such that an irreducible representation of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{C})$ is determined up to conjugacy by the real translation lengths of their images. Since, for each i , each $(g_n)_*(s_i)$ has a representative of uniformly bounded length, each real translation length $l(\rho_n(s_i))$ is also uniformly bounded. Since $\mathfrak{L}(\Gamma)$ is discrete, we may pass to a subsequence, again called $\{\rho_n\}$, so that $l(\rho_n(s_i))$ is constant for all i . Therefore, all representations in this sequence are conjugate to ρ_1 . Let $h_n \in \mathrm{PSL}_2(\mathbb{C})$ be the conjugating element, so that $\rho_n = h_n \rho_1 h_n^{-1}$.

We now observe that U_p is incompressible (i.e. that ρ_1 is injective). First note that, since all the ρ_n are conjugate, if $s \in \text{Ker}(\rho_1)$ is non-trivial, then $s \in \text{Ker}(\rho_n)$ for all n , and $(g_n)_*(s)$ has a representative of uniformly bounded length in X_n . However, since $f_n(g_n(s))$ is inessential, Lemma 7.1 in [3] again guarantees that $\{l_{X_n}(g_n(s))\}$ converges to ∞ , which is a contradiction.

Finally, consider the cover \tilde{N}_U of N corresponding to $\Gamma_U = \pi_1(U)$. The conjugating elements h_n descend to isometries \tilde{h}_n of \tilde{N}_U . Let $\tilde{f}_n : X_n \rightarrow \tilde{N}_U$ denote the lift of f_n to \tilde{N}_U and let $\tilde{x}_n = \tilde{f}_n(x_n)$. Note that there exists $D > 0$ so that if $x \in \tilde{N}_U$ and there exist representatives of $\{\tilde{f}_1(a_1^1), \dots, \tilde{f}_1(a_m^1)\}$ based at x of length at most C , then $d(x, \tilde{x}_1) \leq D$. It then follows that $d(\tilde{x}_n, \tilde{h}_n(\tilde{x}_1)) \leq D$ for all n . Since $\{f_n(X_n)\}$ exits U , one sees that $\tilde{h}_n(U) \cap U$ is non-empty for all large enough n , and that there are infinitely many distinct \tilde{h}_n .

Let N_1 be the quotient of \tilde{N}_U by its full group of orientation-preserving isometries. We have established that the covering map $\pi_U : \tilde{N}_U \rightarrow N_1$ is infinite-to-one on U .

We now recall the Covering Theorem (see [9, Theorem 9.2.2], [4] and Agol [1]):

Covering Theorem. *Suppose that \widehat{M} is a hyperbolic 3-manifold with finitely generated fundamental group and M is a hyperbolic 3-orbifold. If $p : \widehat{M} \rightarrow M$ is an orbifold cover which is infinite-to-one on a neighborhood U of a geometrically infinite end of \widehat{M}_ϵ^0 , then M has finite volume and has a finite manifold cover M' which fibers over the circle such that either*

- a) \widehat{M} is the cover of M' associated to the fiber, or
- b) \widehat{M}_ϵ^0 is a twisted \mathbb{R} -bundle and \widehat{M} is double covered by the manifold \widehat{M}' which is the cover of M' associated to the fiber.

The Covering Theorem immediately implies that Γ_U is the fiber group associated to a 3-manifold which fibers over the circle. If $\Gamma = \Gamma_U$, we are done. If not, we apply the Covering Theorem again to the map $p_U : \tilde{N}_U \rightarrow N$ to conclude that p_U is finite-to-one. Hempel's Finite Index Theorem [5, Theorem 10.5] implies that N_ϵ^0 is a twisted \mathbb{R} -bundle and that the cover p_U is two-to-one.

The following addendum to the Covering Theorem allows us to conclude that Γ is a singular fiber subgroup, in the case that N_ϵ^0 is a twisted \mathbb{R} -bundle.

Proposition 1. *Suppose that \widehat{M} is a geometrically infinite hyperbolic 3-manifold such that \widehat{M}_ϵ^0 is a twisted \mathbb{R} -bundle over a compact surface, and let $p' : \widehat{M}' \rightarrow \widehat{M}$ be a two-fold cover such that $(\widehat{M}')_\epsilon^0$ is an untwisted \mathbb{R} -bundle.*

1) *If there is a finite volume hyperbolic 3-manifold M'' that fibers over the circle with covering $q' : \widehat{M}' \rightarrow M''$ corresponding to the fiber, then \widehat{M} covers a finite volume hyperbolic orbifold M .*

2) *If $p : \widehat{M} \rightarrow M$ is a cover of a finite volume hyperbolic orbifold M , then M has a finite manifold cover $M' = \mathbb{H}^3/G'$ which fibers over the orbifold $S^1/\langle z \mapsto \bar{z} \rangle$ and \widehat{M} is the cover associated to a singular fiber subgroup of G' .*

the quotient $\widehat{M}'/\langle a, b^{2k} \rangle$ fibers over the orbifold $S^1/\langle z \mapsto \bar{z} \rangle$ and that \widehat{M}' is a cover associated to a singular fiber subgroup.) One may then check that $\hat{p} : \widehat{M}' \rightarrow M$ descends to a covering map $\hat{p}' : M' \rightarrow M$. This completes the proof of Proposition 1, which in turn completes the proof of our Main Theorem.

If we combine Proposition 1 with our earlier statement of the Covering Theorem, we get the following slightly sharper version.

Covering Theorem (sharper version). *Suppose that \widehat{M} is a hyperbolic 3-manifold with finitely generated fundamental group and M is a hyperbolic 3-orbifold. If $p : \widehat{M} \rightarrow M$ is an orbifold cover which is infinite-to-one on a neighborhood U of a geometrically infinite end of \widehat{M}_ϵ^0 , then M has finite volume and has a finite manifold cover $M' = \mathbb{H}^3/G'$ such that either*

- (1) M' fibers over the circle and \widehat{M} is the cover associated to a fiber subgroup of G' , or
- (2) M' fibers over the orbifold $S^1/\langle z \mapsto \bar{z} \rangle$ and \widehat{M} is the cover of M' associated to a singular fiber subgroup of G' .

As a corollary of our Main Theorem and the new Covering Theorem we obtain a characterization of geometrically infinite hyperbolic 3-manifolds (with finitely generated fundamental group) which cover finite volume hyperbolic 3-orbifolds.

Corollary. *Let Γ be a finitely generated, torsion-free, geometrically infinite Kleinian group. Its real length spectrum $\mathfrak{L}(\Gamma)$ is discrete if and only if Γ is contained in a cofinite volume Kleinian group.*

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