LENGTH SPECTRA AND DEGENERATION OF FLAT METRICS

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Abstract. In this paper we consider flat metrics (semi-translation structures) on surfaces of finite type. There are two main results. The first is a complete description of when a set of simple closed curves is spectrally rigid, that is, when the length vector determines a metric among the class of flat metrics. Secondly, we give an embedding into the space of geodesic currents and use this to get a boundary for the space of flat metrics. The geometric interpretation is that flat metrics degenerate to *mixed structures* on the surface: part flat metric and part measured foliation.

1. Introduction

From the lengths of all, or some, curves on a surface S, can you identify the metric? To be precise, fix a finite type surface S, denote by $\mathcal{C}(S)$ the set of homotopy classes of closed curves on S, and let $\mathcal{S}(S)$ be the homotopy classes represented by simple closed curves (simply denoted by $\mathcal{C}$ and $\mathcal{S}$ when S is understood). Given an isotopy class of metrics ρ and $\alpha \in \mathcal{C}$, we write $\ell_\rho(\alpha)$ to denote the infimum of lengths of representatives of α in a representative metric for ρ, and we call this the *length of α in ρ* or the *ρ–length of α*. For a set of curves $\Sigma \subset \mathcal{C}$, we define the *(marked) Σ–length spectrum of ρ* to be the length vector, indexed over Σ:

$$\lambda_\Sigma(\rho) = (\ell_\rho(\alpha))_{\alpha\in\Sigma} \in \mathbb{R}^\Sigma.$$

For a family of metrics $\mathcal{G} = \mathcal{G}(S)$, up to isotopy, and a family of curves Σ, we are interested in the problem of deciding when $\lambda_\Sigma(\rho)$ determines ρ. In other words, we ask

Question. *Is the map $\mathcal{G} \to \mathbb{R}^\Sigma$ given by $\rho \mapsto \lambda_\Sigma(\rho)$ an injection?*

If this map is injective, so that $\rho \in \mathcal{G}$ is determined by the lengths of the Σ curves, we say that Σ is *spectrally rigid* over $\mathcal{G}$.

For instance, we may take $\Sigma = \mathcal{S}$, and $\mathcal{G} = \mathcal{T}(S)$, the Teichmüller space of complete finite-area hyperbolic metrics on S (constant curvature -1). Here it is a classical fact due to Fricke that $\mathcal{T}(S) \to \mathbb{R}^\mathcal{S}$ is injective; that is, $\mathcal{S}$ is spectrally rigid over $\mathcal{T}(S)$.

Another natural family of metrics arising in Teichmüller theory consists of those induced by unit-norm quadratic differentials; these are locally flat (isometrically Euclidean) away from a finite number of singular points with cone angles $k\pi$. These are nonpositively curved in the sense of comparison geometry when the surface S is closed, but $k = 1$ is allowed in the case of punctures. We will call these *flat metrics* on S (see Section 2 for a detailed discussion). For example, identifying opposite sides of a regular Euclidean octagon produces a flat metric on a genus two surface. We denote this family of metrics by $\mathsf{Flat}(S)$.

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Theorem 1. *For any finite type surface S, the set of simple closed curves $\mathcal{S}$ is spectrally rigid over* $\mathsf{Flat}(S)$.

Put in other terms, this theorem states that the lengths of simple closed curves determine a quadratic differential up to rotation. Let $\xi(S) = 3g-3+n$ be a measure of the complexity of S, where g is the genus and n is the number of punctures. Then we can compare Theorem 1 to the rigidity over hyperbolic metrics by noting that the dimension of $\mathcal{T}(S)$ is 2ξ, while the dimension of $\mathsf{Flat}(S)$ is $4\xi - 1$.

In fact, we obtain a much sharper version of Theorem 1 which provides a complete answer to the motivating question above for simple closed curves over flat metrics. Let $\mathcal{PMF} = \mathcal{PMF}(S)$ denote Thurston's space of projective measured foliations on S.

Theorem 2. *If $\xi(S) \geq 2$, then $\Sigma \subset \mathcal{S} \subset \mathcal{PMF}$ is spectrally rigid over $\mathsf{Flat}(S)$ if and only if Σ is dense in $\mathcal{PMF}$.*

This theorem stands in contrast to the hyperbolic case, where there are *finite* spectrally rigid sets, as is further discussed in §1.1. We also remark that if $\xi(S) \leq 1$ then it is easy to see that any set of three distinct, primitive curves is spectrally rigid over $\mathsf{Flat}(S)$; see Proposition 16.

Theorem 3. *Suppose $\xi(S) \geq 2$. If $\Sigma \subset \mathcal{S} \subset \mathcal{PMF}$ and $\overline{\Sigma} \neq \mathcal{PMF}$, then there is a deformation family $\Omega_\Sigma \subset \mathsf{Flat}(S)$ for which $\Omega_\Sigma \to \mathbb{R}^\Sigma$ is constant, and such that the dimension of Ω_Σ is proportional to the dimension of $\mathsf{Flat}(S)$ itself.*

In particular, in the closed case, our construction produces $2g - 3$ parameters for deformations, while the dimension of $\mathsf{Flat}(S)$ in this case is $12g - 14$.

Another result needed for the proof of Theorem 2 is a version of Thurston's theorem that the hyperbolic lengths for simple closed curves continuously extends to the space $\mathcal{MF}(S)$ of measured foliations (or laminations) on S. In [5], Bonahon gave a very elegant proof of this (for closed surfaces) based on a unified approach to studying hyperbolic metrics, closed curves and laminations. Bonahon's key idea is to embed $\mathcal{C}(S)$, $\mathcal{T}(S)$ and $\mathcal{MF}(S)$, into the space of geodesic currents $\mathsf{C}(S)$. Our next result extends the theory to flat structures.

Theorem 4. *There is an embedding*

$$\mathsf{Flat}(S) \to \mathsf{C}(S)$$

denoted by $q \mapsto L_q$ so that for $q \in \mathsf{Flat}(S)$ and $\alpha \in \mathcal{C}$, we have $\mathrm{i}(L_q, \alpha) = \ell_q(\alpha)$. Furthermore, after projectivizing, $\mathsf{Flat}(S) \to \mathcal{P}\mathsf{C}(S)$ is still an embedding.

As a consequence, we obtain a continuous homogeneous extension of the flat length function in Corollary 24, making it meaningful to discuss the length of a foliation:

$$\mathsf{Flat}(S) \times \mathcal{MF}(S) \to \mathbb{R}.$$

Remark 5. For the purpose of geodesic currents, punctured surfaces are treated as surfaces with holes; see Section 2.

As $\mathcal{P}\mathsf{C}(S)$ is compact, Theorem 4 provides a compactification of $\mathsf{Flat}(S)$ that is invariant under the action of the mapping class group. Bonahon proved that for closed surfaces, the analogous compactification of $\mathcal{T}(S)$ is precisely the Thurston compactification by projective measured laminations. For the compactification of $\mathsf{Flat}(S)$, we also find a geometric interpretation of the boundary points as *mixed* *structures* on S. A mixed structure is a hybrid of a flat structure on a subsurface (with boundary length zero), and a measured lamination on the complementary subsurface. We view the space of mixed structures as a subspace of $\mathsf{C}(S)$, and thus for any mixed structure η, there is a well-defined intersection number $\mathrm{i}(\eta, \cdot)$. This theory is developed in §6.

Theorem 6. *The closure of $\mathsf{Flat}(S)$ in $\mathcal{P}\mathsf{C}(S)$ is exactly the space $\mathcal{P}\mathsf{Mix}(S)$. That is, for any sequence $\{q_n\}$ in $\mathsf{Flat}(S)$, after passing to a subsequence if necessary, there exists a mixed structure η and a sequence of positive real numbers $\{t_n\}$ so that*

$$\lim_{n\to\infty} t_n \ell_{q_n}(\alpha) = \mathrm{i}(\alpha, \eta).$$

for every $\alpha \in \mathcal{C}$. Moreover, every mixed structure is a limit of a sequence in $\mathsf{Flat}(S)$.

In Sections 6 and 7 we make several other comparisons between this compactification and the Thurston compactification of $\mathcal{T}(S)$.

1.1. **Context: Other spectral rigidity results.** Spectral rigidity of $\mathcal{S}$ over $\mathcal{T}(S)$ was generalized considerably by Otal [24] who showed that $\mathcal{C}$ is spectrally rigid over $\mathcal{G}_-(S)$, the space of isotopy classes of all negatively curved metrics on S. Hersonsky-Paulin [16] generalized this further to show that $\mathcal{C}$ is spectrally rigid over negatively curved cone metrics. This was pushed in a different direction by Croke [9], Fathi [11] and Croke-Fathi-Feldman [8] where it was shown that $\mathcal{C}$ is spectrally rigid for various qualities of nonpositively curved metrics (for more precise statements, see the references).

While these results allow for rather general classes of metrics, the use of all closed curves, not just the simple ones, is essential. Indeed, it follows from a result of Birman-Series [3] that, in general, we should *not* expect $\mathcal{S}$ to be spectrally rigid for an arbitrary class of negatively curved metrics since simple closed curves miss most of the surface (see §7).

We saw above in Theorem 2 that a set of curves must be dense in the sphere $\mathcal{PMF}$ in order to be spectrally rigid over $\mathsf{Flat}(S)$. This stands in contrast with the situation for hyperbolic metrics, where it is known that there are finite spectrally rigid sets (see [14, 15, 28] for a discussion of optimal sets of curves). In this regard, $\mathsf{Flat}(S)$ has more in common with Outer space, $\mathsf{CV}(F_n)$. The Culler–Vogtmann Outer space, built to study the group $Out(F_n)$ in analogy to the relationship between $\mathcal{T}(S)$ and the mapping class group, consists of metric graphs X equipped with a isomorphisms $F_n \to \pi_1(X)$ (under the equivalence relation of graph isometries which respect the isomorphism up to conjugacy). Recycling notation suggestively, let $\mathcal{C}$ denote the set of conjugacy classes of nontrivial elements of F_n. Given an element $X \in \mathsf{CV}(F_n)$, and a conjugacy class $\alpha \in \mathcal{C}$, we write $\ell_X(\alpha)$ for the minimal-length representative of α in X. We can define a length spectrum just as above:

$$\lambda_\Sigma(X) = (\ell_X(\alpha))_{\alpha\in\Sigma} \in \mathbb{R}^\Sigma$$

for $X \in \mathsf{CV}(F_n)$ and $\Sigma \subset \mathcal{C}$. Accordingly, we say that Σ is spectrally rigid over $\mathsf{CV}(F_n)$ if $X \mapsto \lambda_\Sigma(X)$ is injective.

The full set $\mathcal{C}$ is spectrally rigid over $\mathsf{CV}(F_n)$ [2, 10]. However, Smillie and Vogtmann (expanding on a similar result of Cohen, Lustig and Steiner [7]) showed that no finite subset $\Sigma \subset \mathcal{C}$ is spectrally rigid over Outer space (or even the reduced Outer space) by finding a $(2n-5)$-parameter family of graphs over which λ_Σ is constant [29]. Thus, Theorem 3 is the analog for $\mathsf{Flat}(S)$ of the Smillie–Vogtmann result. Our proof of Theorem 3 adapts the key idea from Smillie–Vogtmann to surfaces by appealing to Thurston's theory of train tracks; see §4. Thus, from a length spectral rigidity point of view, flat metrics might be said to resemble metric graphs more closely than hyperbolic metrics.

Finally, we briefly consider unmarked inverse spectral problems for the metrics in $\mathsf{Flat}(S)$. Kac memorably asked in 1966 whether one can "hear the shape of a drum," or determine a planar region by the eigenvalues of its Laplacian. Sunada's work in the 1980s established a means of generating examples of hyperbolic surfaces which are not only isospectral with respect to their Laplacians, but iso-length-spectral as well. That is, let the *unmarked length spectrum* be the nondecreasing sequence of numbers

$$\Lambda_{\mathcal{C}}(\rho) = \{\ell_\rho(\gamma_1) \le \ell_\rho(\gamma_2) \le \cdots\}_{\gamma_i \in \mathcal{C}},$$

appearing as lengths of closed curves on S, counted with multiplicity. Sunada's construction produces a supply of examples of hyperbolic metrics m, m' such that $\Lambda_{\mathcal{C}}(m) = \Lambda_{\mathcal{C}}(m')$. In § 7.3, we remark that the Sunada construction carries over to our flat metrics in the same way.

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2. Preliminaries: Flat structures and geodesic currents

In this section, we will briefly describe the background and preliminary material on Teichmüller theory, semi-translation surfaces, flat metrics, and Bonahon's theory of geodesic currents. We refer the reader to [4], [5], [12], [25], and [30].

Unless otherwise stated, in what follows S is a finite-type surface. That is, S is obtained from a closed surface $\hat{S}$ by removing a finite set $P \subset \hat{S}$ of marked points. The genus g and number of punctures $n = |P|$ determine the complexity

$$\xi = \xi(S) = 3g - 3 + n.$$

Recall that Teichmüller space $\mathcal{T}(S)$ is homeomorphic to a ball of dimension 2ξ.

2.1. Quadratic differentials and semi-translation structures.

By a *quadratic differential* on S we mean a complex structure on $\hat{S}$ together with an integrable meromorphic quadratic differential. The quadratic differential is allowed to have poles of degree one on marked points and is assumed to be holomorphic on S. The space of all quadratic differentials, defined up to isotopy, is denoted $\mathcal{Q}(S)$. A point of $\mathcal{Q}(S)$ will be denoted q, with the underlying complex structure implicit in the notation. Reading off the complex structures, we obtain a projection to the Teichmüller space

$$\pi : \mathcal{Q}(S) \to \mathcal{T}(S).$$

This projection is canonically identified with the cotangent bundle to $\mathcal{T}(S)$, hence $\mathcal{Q}(S)$ has a real dimension of 4ξ.

Integrating the square root of a nonzero quadratic differential q in a small neighborhood of a point where q is nonzero produces *natural coordinates* ζ on S in which $q = d\zeta^2$. The set of all natural coordinates is an atlas on the complement of the zeros of q for which the transition functions are given by $z \mapsto \pm z + c$ for $c \in \mathbb{C}$ (called *semi-translations*). The Euclidean metric is preserved by these transition functions and so pulls back to a Euclidean metric on the complement of the zeros of q in S. The integrability of q implies that the metric has finite total area.

The completion of the metric is obtained by replacing the zeros of q as well as the points P to obtain the surface $\hat{S}$. If q has a zero of order p at one of the completion points, then there is a cone singularity with cone angle $(2+p)\pi$ (here a pole at a point of P is thought of as a zero of order -1). Thus the metric on S is locally CAT(0), although the metric on $\hat{S}$ may not be. We also use q to denote the completed metric on $\hat{S}$.

A *semi-translation structure* is a locally CAT(0) Euclidean cone metric on S, whose completion is $\hat{S}$, together with an atlas defining the metric away from the cone points whose transition functions are semi-translations. The atlas determines (and is determined by) a preferred vertical direction. Given a semi-translation structure, there is a unique complex structure and integrable holomorphic quadratic differential for which the charts in the atlas are natural coordinates. This determines a bijection between the set of nonzero quadratic differentials and the set of semi-translation structures on S, which we use to identify the two spaces. The Teichmüller metric is determined by the co-norm on $\mathcal{Q}(S)$ given by the area of the associated semi-translation structure on S. The unit cotangent space, $\mathcal{Q}^1(S)$, is thus precisely the set of unit-area semi-translation structures on S.

A semi-translation structure can also be described combinatorially as a collection of (possibly punctured) polygons in the Euclidean plane with sides identified in pairs by an isometry which is the restriction of a semi-translation.

The group $\mathrm{SL}_2(\mathbb{R})$ acts naturally on the space of quadratic differentials by $\mathbb{R}$–linear transformation on the natural coordinates. The geodesics in the Teichmüller metric are precisely projections to $\mathcal{T}(S)$ of orbits of the $\mathrm{SL}_2(\mathbb{R})$ diagonal on an initial quadratic differential q_0:

$$\gamma(t) = \left\{ \pi(A_t.q_0) : A_t = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix} \right), t \in \mathbb{R} \right\}.$$

The *Teichmüller disk* $\mathbb{H}_q$ of a quadratic differential q is the projection to $\mathcal{T}(S)$ of its entire $\mathrm{SL}_2(\mathbb{R})$ orbit; it is an isometrically embedded copy of the hyperbolic plane of curvature -4.

We let $p : \widetilde{S} \to S$ denote the universal covering of S with $\pi_1(S)$ acting by covering transformations. The metric q pulls back to a metric $\tilde{q} = p^*(q)$ on $\widetilde{S}$ which is locally CAT(0). When S is a closed surface, $(\widetilde{S}, \tilde{q})$ is a complete geodesic CAT(0) space. If S has punctures, then $(\widetilde{S}, \tilde{q})$ is incomplete, and we write $(\bar{S}, \tilde{q})$ for the completion, which is a geodesic CAT(0) space. The covering $p : \widetilde{S} \to S$ can be extended to the completions which we also denote $p : \bar{S} \to \hat{S}$. This extension can be be viewed as a branched cover, infinitely branched over P, and we let $\widetilde{P}$ denote the preimage of P in $\bar{S}$.

2.2. **Measured foliations and measured laminations.** We write $\mathcal{MF} = \mathcal{MF}(S)$ and $\mathcal{PMF} = \mathcal{PMF}(S)$ to denote the space of (measure classes of) measured foliations, and projective measured foliations, on S, respectively. A curve $\alpha \in \mathcal{S}$ determines a measured foliation with all nonsingular leaves closed and homotopic to α. When convenient, we use this to view $\mathbb{R}_+ \times \mathcal{S}$ and $\mathcal{S}$ as subsets of $\mathcal{MF}$ and

$\mathcal{PMF}$, respectively. We also write

$$\mathrm{i} : \mathcal{MF} \times \mathcal{MF} \to \mathbb{R}$$

for Thurston's geometric intersection number. This is the unique homogeneous continuous extension of the usual geometric intersection number of $\mathcal{S} \times \mathcal{S}$, via the inclusions described above.

The vertical foliation for a nonzero quadratic differential $q \in \mathcal{Q}(S)$ is given by $|\operatorname{Re}(\sqrt{q})|$. Let ν_q^θ be the foliation $|\operatorname{Re}(e^{i\theta}\sqrt{q})|$, for $\theta \in \mathbb{R}\mathrm{P}^1$, so that the vertical foliation of q is $\nu_q := \nu_q^0$. By letting

$$\mathcal{MF}(q) := \{\ t \cdot \nu_q^\theta\ :\ \theta \in \mathbb{R}\mathrm{P}^1, t \in \mathbb{R}_+\},$$

we obtain the set of all measured foliations which are straight in some direction on q, with measure proportional to Euclidean distance between leaves. We write $\mathcal{PMF}(q)$ for the projectivization of $\mathcal{MF}(q)$.

It will be useful to pass back and forth between measured foliations and measured laminations. We denote the space of measured laminations by $\mathcal{ML}$ and the projective measured laminations by $\mathcal{PML}$. We identify $\mathcal{MF} = \mathcal{ML}$ and $\mathcal{PMF} = \mathcal{PML}$ in the natural way extending the canonical inclusions of $\mathcal{S}$. See [19] for an explicit procedure for constructing laminations from foliations.

2.3. **Flat structures.** Quadratic differentials that represent the same metric differ only by a rotation. Accordingly, the space of *flat metrics* is defined as

$$\mathsf{Flat}(S) = \mathcal{Q}^1(S) \Big/ q \sim e^{i\theta} q.$$

Equivalently, an element of $\mathsf{Flat}(S)$ is a Euclidean cone metric on S which is locally CAT(0), with holonomy in $\{\pm I\}$, completion $\hat{S}$, and total area one. This is almost identical to the notion of a quadratic differential, but there is one missing piece of data, namely the notion of a preferred vertical which is determined by the atlas of natural coordinates. We write q to denote a point in $\mathcal{Q}^1(S)$ or the associated equivalence class in $\mathsf{Flat}(S)$. Note that $\mathcal{MF}(q)$ and $\mathcal{PMF}(q)$ are well-defined for $q \in \mathsf{Flat}(S)$. Also, each Teichmüller disk $\mathbb{H}_q$ embeds into $\mathsf{Flat}(S)$, and in fact, $\mathsf{Flat}(S)$ is foliated by Teichmüller disks.

2.4. **Geodesics.** Let q be a quadratic differential on S and $(\bar{S}, \tilde{q})$ the metric completion of the metric pulled back to the universal cover as described above. Every curve $\alpha \in \mathcal{C}$ has a *q–geodesic representative* in the following sense; for a map $\alpha\colon S^1 \to S$ from the unit circle to S, there is an isometry $\tilde{\alpha}_q\colon \mathbb{R} \to (\bar{S}, \tilde{q})$ such that a subgroup of $\pi_1(S)$ corresponding to the curve α preserves the image $\tilde{\alpha}_q(\mathbb{R})$. The projection of this to $\hat{S}$ is the q–geodesic representative of α and we denote it by α_q. (See [27] for more details.) We call the isometry $\tilde{\alpha}_q$, or any $\pi_1(S)$–translate of it a *lift* of α_q.

The geodesic representative of α is unique (up to parameterization), except when there are a family of parallel geodesic representatives foliating a flat cylinder. The geodesic representative of a simple closed curve need not be simple and the geodesic representative of different curves may not be different. For example, curves that go around a puncture different number of times can have the same geodesic representative that passes through the puncture (the number of times a curve goes around the puncture is not detectable from the geodesic representative). However, for every curve α, there is always a sequence of representatives of the homotopy class of α in S converging uniformly to α_q.

When S is a punctured surface, we will also be interested in homotopy classes of essential proper paths in S. These are paths $\alpha\colon I \to \hat{S}$, defined on some closed interval I, for which the interior of I is mapped to S and the endpoints are mapped to P. Here, two such paths are homotopic if there is a homotopy relative to the endpoints so that throughout the homotopy the interior of I is mapped to S. We denote the set of all homotopy classes of essential curves and paths by $\mathcal{C}'(S)$, which is equal to $\mathcal{C}(S)$ if S is closed. Every element of $\mathcal{C}'(S)$ has a unique geodesic representative, which we view as the projection of an isometry $\tilde{\alpha}_q\colon I \to (\bar{S}, \tilde{q})$ to $\hat{S}$, and is again denoted by α_q. Again, α_q is a uniform limit of representatives of the homotopy class of α.

When a curve α has non-unique geodesic representatives that foliate a cylinder, we say α is a *cylinder curve* and we define the *cylinder set* of q, denoted by $\mathsf{cyl}(q)$, to be the set of all cylinder curves with respect to q.

A *saddle connection* is a geodesic segment whose endpoints are (not necessarily distinct) singularities or points of P, and which has no singularities in its interior. When $\alpha \in \mathcal{C}'(S)$ is not a cylinder curve, the (unique) geodesic representative is made up of concatenations of saddle connections. (In fact, each boundary component of a cylinder is a union of saddle connections, so even cylinder curves have representatives of this form.) If we write this concatenation as

$$\alpha_q = \alpha^1 \cdots \alpha^k$$

and let r_j denote the Euclidean length of α^j, then $\ell_q(\alpha)$ is just $r_1 + \cdots + r_k$.

If we view q as a quadratic differential (and not just as a flat structure), then each α_j makes some angle θ_j with the horizontal direction.

Lemma 7. *For all $q \in \mathcal{Q}^1(S)$ and $\alpha \in \mathcal{C}'(S)$, we have*

$$\ell_q(\alpha) = \frac{1}{2}\int_0^\pi \mathrm{i}(\nu_q^\theta, \alpha) d\theta.$$

Proof. This is a computation:

$$\begin{aligned}\int_0^\pi \mathrm{i}(\nu_q^\theta, \alpha)\, d\theta &= \int_0^\pi \left(\sum_{j=1}^k \int_{\alpha_j} |\operatorname{Re}(e^{i\theta}\sqrt{q}\,)|\right) d\theta \\ &= \sum_{j=1}^k \int_0^\pi r_j |\cos(\theta+\theta_j)|\, d\theta = \sum_{j=1}^k 2r_j \;=\; 2\ell_q(\alpha).\end{aligned}$$

□

While the q–geodesics α_q and β_q are not necessarily embedded or transverse, they do meet minimally in a certain sense. Namely, appealing to the CAT(0) structure, we first note that any two lifts $\widetilde{\alpha}_q$ and $\widetilde{\beta}_q$ meet in a point, in a geodesic segment, or they are disjoint. If the endpoints at infinity of $\widetilde{\alpha}_q$ and $\widetilde{\beta}_q$ nontrivially link, then we call these intersections *essential intersections*. It follows that $\mathrm{i}(\alpha, \beta)$ is the number of $\pi_1(S)$–orbits of essential intersections over all lifts of α_q and β_q.

Remark 8. We make an elementary but very useful observation that identifies the geodesics in a flat metric q. First consider the case that S is closed. Given a representative of α built as a concatenation of saddle connections $\alpha^1 \cdots \alpha^k$, a necessary and sufficient condition for this to be a q-geodesic is that the angles between successive α^i measure at least π on both sides. When P is nonempty, we need to modify this slightly. If $\alpha^1 \cdots \alpha^k$ is a representative of α in $\hat{S}$ (that is, it is a limit of representatives of α in S), then a lift of this representative to $\bar{S}$ (by which we mean a lift which is a limit of lifts of representatives of α), subtends an angle at least π at each point in $\widetilde{P}$. Note that there is a unique, well-defined, finite angle at any point of $\widetilde{P}$ which the lift meets.

2.5. **Geodesic currents.** For this discussion, we first restrict to the closed case ($P = \emptyset$). We fix any geodesic metric g on S. We can pull back this metric to the universal covering $p : \tilde{S} \to S$, so that the covering group action of $\pi_1(S)$ on $\tilde{S}$ is by isometries. We let $\tilde{S}_\infty$ denote the Gromov boundary of $\tilde{S}$, making $\tilde{S} \cup \tilde{S}_\infty$ into a closed disk. This compactification is independent of the choice of metric (in the sense that a different choice of metric gives an alternate compactification for which the identity extends to a homeomorphism of the boundary circles).

We consider the space

$$G(\tilde{S}) = (\tilde{S}_\infty \times \tilde{S}_\infty \setminus \Delta) \Big/ (x, y) \sim (y, x).$$

With respect to our metric, this is precisely the space of unoriented biinfinite geodesics in $\tilde{S}$ up to bounded Hausdorff distance. We endow $G(\tilde{S})$ with the diagonal action of $\pi_1(S)$.

A *geodesic current on* S is a $\pi_1(S)$–invariant Radon measure on $G(\tilde{S})$. The set of all geodesic currents is made into a (metrizable) topological space by imposing the weak* topology, and we denote this space $\mathsf{C}(S)$. The associated space of *projective currents* is the quotient of the space of nonzero currents by positive real scalar multiplication, and we denote it $\mathcal{P}\mathsf{C}(S)$.

The simplest examples of geodesic currents are defined by closed curves $\alpha \in \mathcal{C}$ as follows. Given such a curve α, we first realize it by a geodesic representative (with respect to our fixed metric). The preimage $p^{-1}(\alpha)$ in $\tilde{S}$ determines a discrete subset of $G(\tilde{S})$ (independent of the metric), and to this we can associate a Dirac measure on $G(\tilde{S})$, for which $\pi_1(S)$–invariance follows from the invariance of $p^{-1}(\alpha)$. This injects the set $\mathcal{C}$ into $\mathsf{C}(S)$, and we will thus view $\mathcal{C}$ as a subset of $\mathsf{C}(S)$ when convenient. While these are very special types of geodesic currents, the set of positive real multiples of all curves is in fact dense in $\mathsf{C}(S)$, as shown in [5].

In [4], Bonahon constructs a continuous extension for the intersection number to all currents.

Theorem 9 (Bonahon)**.** *The geometric intersection number* $\mathrm{i} : \mathcal{C}(S) \times \mathcal{C}(S) \to \mathbb{R}$ *has a continuous, bilinear extension*

$$\mathrm{i} : \mathsf{C}(S) \times \mathsf{C}(S) \to \mathbb{R}.$$

Moreover, in [24], Otal proved that i and $\mathcal{C}$ can be used to separate points:

Theorem 10 (Otal)**.** *Given* $\mu_1, \mu_2 \in \mathsf{C}(S)$*,* $\mu_1 = \mu_2$ *if and only if* $\mathrm{i}(\mu_1, \alpha) = \mathrm{i}(\mu_2, \alpha)$ *for all* $\alpha \in \mathcal{C}$*.*

From this, one can easily deduce a convergence criterion, and also define a metric on the space of currents which will be convenient for our purposes.

Theorem 11. *A sequence* $\mu_k \in \mathsf{C}(S)$ *converges to* $\mu \in \mathsf{C}(S)$ *if and only if*

$$\lim_{k \to \infty} \mathrm{i}(\mu_k, \alpha) = \mathrm{i}(\mu, \alpha),$$

for all $\alpha \in \mathcal{C}$. Furthermore, there exist $t_\alpha \in \mathbb{R}_+$ for each $\alpha \in \mathcal{C}$ so that

$$d(\mu_1, \mu_2) = \sum_{\alpha \in \mathcal{C}} t_\alpha \big| \mathrm{i}(\mu_1, \alpha) - \mathrm{i}(\mu_2, \alpha) \big|$$

defines a proper metric on $\mathsf{C}(S)$ which is compatible with the weak topology.*

Before we prove this theorem, we recall one further fact due to Bonahon [5] which we will need. We say that a geodesic current ν is *binding* if for every $(x, y) \in G(\tilde{S})$, there is an (x', y') in the support of ν such that (x, y) and (x', y') link in $\tilde{S}_\infty$. With respect to any fixed metric, this is equivalent to requiring that every biinfinite geodesic in $\tilde{S}$ intersects some geodesic in the support of ν. It follows, as discussed by Bonahon, that any binding current and any nonzero current have positive intersection number. As an example, any filling curve or union of curves determines a binding current.

Proposition 12 (Bonahon)**.** *If ν is a binding geodesic current and $R > 0$, then the set*

$$\{\mu \in \mathsf{C}(S) \mid \mathrm{i}(\mu, \nu) \leq R\}$$

is a compact set. Consequently, the set

$$\left\{ \frac{\mu}{\mathrm{i}(\mu, \nu)} \,\middle|\, \mu \in \mathsf{C}(S) \setminus \{0\} \right\}$$

is compact, and hence so is $\mathcal{P}\mathsf{C}(S)$.

Proof of Theorem 11. Continuity of i implies $\mathrm{i}(\mu_k, \alpha) \to \mathrm{i}(\mu, \alpha)$ for all $\alpha \in \mathcal{C}$ if $\mu_k \to \mu$. To prove the other direction, assume $\mathrm{i}(\mu_k, \alpha) \to \mathrm{i}(\mu, \alpha)$ for all $\alpha \in \mathcal{C}$. In particular, if we let $\alpha_0 \in \mathcal{C}$ be a filling curve (so the associated current is binding), then $\mathrm{i}(\mu_k, \alpha_0), \mathrm{i}(\mu, \alpha_0) \leq R$ for some $R > 0$. So, $\{\mu_k\} \cup \{\mu\}$ is contained in some compact set by Proposition 12.

Since $\mathsf{C}(S)$ is metrizable, it follows that there is a convergent subsequence $\mu_{k_n} \to \mu'$ for some $\mu' \in \mathsf{C}(S)$. Continuity of i implies that $\mathrm{i}(\mu, \alpha) = \mathrm{i}(\mu', \alpha)$ for all α, and so Theorem 10 guarantees that $\mu = \mu'$. Since this is true for any convergent subsequence of $\{\mu_k\}$ it follows that $\mu_k \to \mu$. This completes the proof of the first statement of the theorem.

To build the metric we must first find the numbers $\{t_\alpha\}$. For this, we observe that for any $\mu \in \mathsf{C}(S)$ and fixed choice of a filling curve α_0, the numbers

$$\left\{ \frac{\mathrm{i}(\mu, \alpha)}{\mathrm{i}(\alpha_0, \alpha)} \right\}_{\alpha \in \mathcal{C}} = \left\{ \mathrm{i}\left(\mu,\ \frac{\alpha}{\mathrm{i}(\alpha_0, \alpha)}\right) \right\}_{\alpha \in \mathcal{C}}$$

are uniformly bounded. This follows from the fact that the set of currents

$$\left\{ \frac{\alpha}{\mathrm{i}(\alpha_0, \alpha)} \right\}_{\alpha \in \mathcal{C}}$$

is precompact by Proposition 12.

Now we enumerate all closed curves $\alpha_0, \alpha_1, \alpha_2, ... \in \mathcal{C}$ (α_0 still denoting our filling curve). Set $t_k = t_{\alpha_k} = 1/(2^k\, \mathrm{i}(\alpha_0, \alpha_k))$. It follows that

$$\sum_{k=0}^{\infty} t_k\, \mathrm{i}(\mu, \alpha_k) = \sum_{k=0}^{\infty} \frac{1}{2^k}\ \mathrm{i}\left(\mu, \frac{\alpha_k}{\mathrm{i}(\alpha_0, \alpha_k)}\right)$$

converges and hence the series for d given in the statement of the proposition converges. Symmetry and the triangle inequality are immediate, and positivity follows from Theorem 10. The fact that the topology agrees with the weak* topology is a consequence of the first part of the Theorem and the fact that $\mathsf{C}(S)$ is metrizable (hence first countable, so determined by its convergent sequences).

Finally, we verify that the metric is proper. Proposition 12 implies that for any binding current $\nu \in \mathsf{C}(S)$, the set

$$A = \left\{ \frac{\mu}{\mathrm{i}(\mu, \nu)} \,\Big|\, \mu \in \mathsf{C}(S) \setminus \{0\} \right\}$$

is compact. Since d is continuous, the distance from 0 to any point of A is bounded above by some $R > 0$ and below by some $r > 0$. Furthermore, for any $\mu \in \mathsf{C}(S)$ and $t \in \mathbb{R}_+$, we have

$$d(t\mu, 0) = t \cdot d(\mu, 0).$$

Hence, the compact set

$$A' = \{t\mu \,|\, \mu \in A\,,\, t \in [0, 1]\}$$

is contained in the ball of radius R and contains the ball of radius r. From this and the preceding equation, it follows that for any $\rho > 0$, the closed ball of radius $\rho > 0$ about 0 is a compact set. That is, d is a proper metric. □

2.6. **Punctured surfaces.** The situation for punctured surfaces requires more care. First, we replace all punctures by holes, so that we may uniformize S by a convex cocompact hyperbolic surface. That is, we give S a complete hyperbolic metric (of infinite area) so that S contains a compact, *convex core* which we denote $\mathrm{core}(S)$. To describe $\mathrm{core}(S)$ concretely, first consider the universal covering $\tilde{S} \to S$ (with $\tilde{S}$ isometric to the hyperbolic plane) together with the isometric action of $\pi_1(S)$ by covering transformations. We denote the limit set of the action on the circle at infinity of $\tilde{S}$ by $\Lambda \subset \tilde{S}_\infty$. The convex hull of Λ in $\tilde{S}$ is a closed, $\pi_1(S)$–invariant set which we denote $\mathrm{hull}(\Lambda)$, and the quotient by $\pi_1(S)$ is precisely $\mathrm{core}(S)$. The inclusion $\mathrm{core}(S) \subset S$ is a homotopy equivalence and the convex cocompactness means that $\mathrm{core}(S)$ is compact. Let $G(\mathrm{hull}(\Lambda))$ denote the space of geodesics in $\widetilde{S}$ with both endpoints in Λ. Thus,

$$G(\mathrm{hull}(\Lambda)) \cong (\Lambda \times \Lambda - \Delta)/(x, y) \sim (y, x).$$

A geodesic current on S is now defined to be a $\pi_1(S)$–invariant Radon measure on $G(\mathrm{hull}(\Lambda))$. Equivalently, we are considering $\pi_1(S)$–invariant measures on $G(\tilde{S})$ for which the support consists of geodesics that project entirely into $\mathrm{core}(S)$. We use the same notation as before and denote the space of currents on S by $\mathsf{C}(S)$, endowed with the weak* topology. Bonahon also proves that the associated projective space $\mathcal{P}\mathsf{C}(S)$ is compact and that the geometric intersection number on closed curves extends continuously to a symmetric bilinear function

$$\mathrm{i} : \mathsf{C}(S) \times \mathsf{C}(S) \to \mathbb{R}.$$

In this setting, the conclusion of Theorem 10 is not true: the geodesic currents associated to boundary curves have zero intersection number with every geodesic current. We remedy this as follows.

First suppose that $\alpha : \mathbb{R} \to S$ is a proper biinfinite geodesic. If we let $\tilde{\alpha} : \mathbb{R} \to \widetilde{S}$ denote a lift of α, then both endpoints limit to points in $\widetilde{S}_\infty - \Lambda$. As such, the set of all geodesics in $G(\text{hull}(\Lambda))$ which transversely intersect $\tilde{\alpha}(\mathbb{R})$ is a compact set which we denote $A_{\tilde{\alpha}}$. Given $\mu \in \mathsf{C}(S)$, we define

$$\mathrm{i}(\mu, \alpha) = \mu(A_{\tilde{\alpha}}).$$

Lemma 13. *For any proper biinfinite geodesic $\alpha : \mathbb{R} \to S$, the function*

$$\mathsf{C}(S) \to \mathbb{R}$$

given by $\mu \mapsto \mathrm{i}(\mu, \alpha)$ is continuous and depends only on the proper homotopy class of $\alpha \in \mathcal{C}'(S)$.

Proof. The $\pi_1(S)$–equivariance of μ shows that $\mathrm{i}(\mu, \alpha)$ is independent of the chosen lift $\tilde{\alpha} : \mathbb{R} \to \widetilde{S}$. Moreover, a proper homotopy α_t of α lifts to a homotopy $\tilde{\alpha}_t$ for which no endpoint ever meets Λ. It follows that $A_{\tilde{\alpha}_t} = A_{\tilde{\alpha}}$ for all t and so $\mathrm{i}(\mu, \alpha)$ depends only on the homotopy class $\alpha \in \mathcal{C}'(S)$.

All that remains to prove is continuity. Suppose $\mu_k \to \mu$ in $\mathsf{C}(S)$. Then since the characteristic function χ of $A_{\tilde{\alpha}}$ is a compactly supported continuous function, it follows that

$$\mathrm{i}(\mu_k, \alpha) = \int_{G(\text{hull}(\Lambda))} \chi d\mu_k \to \int_{G(\text{hull}(\Lambda))} \chi d\mu = \mathrm{i}(\mu, \alpha)$$

as required. □

Appealing to the closed case, this provides us with enough intersection numbers to prove the analog of Theorem 10 in the present setting.

Let DS be the double of core(S) over its boundary which naturally inherits a hyperbolic metric from that on core(S). We consider core(S) as isometrically embedded in DS. The cover of DS associated to $\pi_1(\text{core}(S)) < \pi_1(DS)$ is canonically isometric to S, and we can identify the two surfaces writing $S \to DS$ for this cover. Thus we have a canonical identification of universal coverings $\tilde{S} = \widetilde{DS}$. The action of $\pi_1(S)$ on $\tilde{S}_\infty$ is the restriction to $\pi_1(S) < \pi_1(DS)$ of the action of $\pi_1(DS)$. Any geodesic current $\mu \in \mathsf{C}(S)$ can be extended to a current in $\mathsf{C}(DS)$, which we also denote μ, by pushing the measure around via coset representatives of $\pi_1(S) < \pi_1(DS)$ making it $\pi_1(DS)$–equivariant.

This defines an injection $\mathsf{C}(S) \to \mathsf{C}(DS)$, and it is straightforward to check that this is an embedding. It follows from Bonahon's construction of the intersection number function that i on $\mathsf{C}(S)$ is just the restriction, via this embedding, of i on $\mathsf{C}(DS)$. If α is any closed geodesic on DS then there are a finite (possibly zero) number of lifts of α to the cover $S \to DS$ that nontrivially meet core(S), and we denote these

$$\alpha^1, \cdots, \alpha^k : \mathbb{R} \to S.$$

If the image is entirely contained in core(S), then there is only one lift, and it covers a closed geodesic. Otherwise, $\alpha^1, \cdots, \alpha^k$ is a union of proper geodesics in S. An inspection of Bonahon's definition of i reveals that for any $\mu \in \mathsf{C}(S)$

$$\mathrm{i}(\mu, \alpha) = \sum_{i=1}^{k} \mathrm{i}(\mu, \alpha^i).$$

We can now prove the required analog of Theorem 10.

Theorem 14. *Given $\mu_1, \mu_2 \in \mathsf{C}(S)$, $\mu_1 = \mu_2$ if and only if $\mathrm{i}(\mu_1, \alpha) = \mathrm{i}(\mu_2, \alpha)$ for all $\alpha \in \mathcal{C}'(S)$.*

Proof. If $\mu_1 \neq \mu_2$, we must find $\alpha \in \mathcal{C}'(S)$ so that $\mathrm{i}(\mu_1, \alpha) \neq \mathrm{i}(\mu_2, \alpha)$. By Theorem 10, there exists $\alpha \in \mathcal{C}(DS)$ so that $\mathrm{i}(\mu_1, \alpha) \neq \mathrm{i}(\mu_2, \alpha)$. If α is contained in core(S), then $\alpha \in \mathcal{C}(S) \subset \mathcal{C}'(S)$ and we are done. Otherwise, let $\alpha^1, ..., \alpha^k \in \mathcal{C}'(S)$ be the lifts as described above. Then

$$\sum_{i=1}^{k} \mathrm{i}(\mu_1, \alpha^i) = \mathrm{i}(\mu_1, \alpha) \neq \mathrm{i}(\mu_2, \alpha) = \sum_{i=1}^{k} \mathrm{i}(\mu_2, \alpha^i).$$

But then $\mathrm{i}(\mu_1, \alpha^i) \neq \mathrm{i}(\mu_2, \alpha^i)$ for some i, completing the proof. □

We also easily obtain a version of Theorem 11.

Theorem 15. *A sequence $\{\mu_k\} \in \mathsf{C}(S)$ converges to $\mu \in \mathsf{C}(S)$ if and only if*

$$\lim_{k \to \infty} \mathrm{i}(\mu_k, \alpha) = \mathrm{i}(\mu, \alpha)$$

for all $\alpha \in \mathcal{C}'(S)$. Furthermore, there exist $t_\alpha \in \mathbb{R}_+$ for each $\alpha \in \mathcal{C}'(S)$ so that

$$d(\mu_1, \mu_2) = \sum_{\alpha \in \mathcal{C}'(S)} t_\alpha \big| \mathrm{i}(\mu_1, \alpha) - \mathrm{i}(\mu_2, \alpha) \big|$$

defines a proper metric on $\mathsf{C}(S)$ which is compatible with the weak topology.* □

Proof. Although we do not have Proposition 12, this proposition applied to DS implies that if $\alpha_0 \in \mathcal{C}(DS)$ is a filling curve, then the associated proper geodesics $\alpha^1, ..., \alpha^k \in \mathcal{C}'(S)$ have the property that

$$A = \left\{ \frac{\mu}{\sum_j \mathrm{i}(\mu, \alpha^j)} \,\Big|\, \mu \in \mathsf{C}(S) \setminus 0 \right\}$$

is compact. The proof continues as for Theorem 11. □

3. Spectral rigidity for simple closed curves

This section is devoted to the proof of Theorem 1. We begin by considering the case of the torus. This is not a step in proving the theorem, but the proof illustrates a useful principle used later, and also shows that Theorem 2 is false for tori (and similarly for once-punctured tori and four-times-punctured spheres).

Proposition 16. *The lengths of any three distinct primitive closed curves determine a flat metric on the torus.*

Proof. The Teichmüller space of unit-area flat tori is the hyperbolic plane. Within this parameter space, prescribing the length of a given curve picks out a horocycle in $\mathbb{H}$. The intersection of two horocycles is at most two points, so by choosing three arbitrary curves, we can determine the flat metric on a torus by their lengths. □

The proof of spectral rigidity for simple closed curves follows from a series of lemmas. The first states that $\lambda_{\mathcal{S}}(q)$ determines $\mathsf{cyl}(q)$.

Lemma 17. *For $\alpha \in \mathcal{S}$ and $q \in \mathsf{Flat}(S)$, $\alpha \notin \mathsf{cyl}(q)$ if and only if there exists $\beta \in \mathcal{S}$ with $\mathrm{i}(\alpha, \beta) \neq 0$ so that the following condition holds:*

(1) $$\ell_q(T_\alpha(\beta)) - \ell_q(\beta) = \ell_q(\alpha) \cdot \mathrm{i}(\alpha, \beta).$$

Proof. First, suppose $\alpha \in \mathsf{cyl}(q)$. Fix any β with $\mathrm{i}(\alpha,\beta) \neq 0$. We must show that α, β, q do not satisfy (1).

Let α_q denote a q–geodesic representative contained in the interior of its Euclidean cylinder neighborhood C and let β_q denote a q–geodesic representative of β. Either β_q is obtained by traversing a finite number of saddle connections or else is itself a cylinder curve (defining a different cylinder than α) and contains no singularities. It follows that $\beta_q \cap C$ consists of finitely many straight arcs connecting one boundary component of C to the other and the number of transverse intersections of α_q and β_q is $\mathrm{i}(\alpha,\beta)$.

We can construct a representative of $T_\alpha(\beta)$ as follows. An arc δ of the intersection $\delta \subset \beta_q \cap C$ is cut by α_q into two arcs $\delta = \delta_0 \cup \delta_1$. To obtain $T_\alpha(\beta)$, surger in a copy of α_q traversed positively; see Figure 1. Observe that this is necessarily not a geodesic representative since it makes an angle less than π at each of the surgery points.

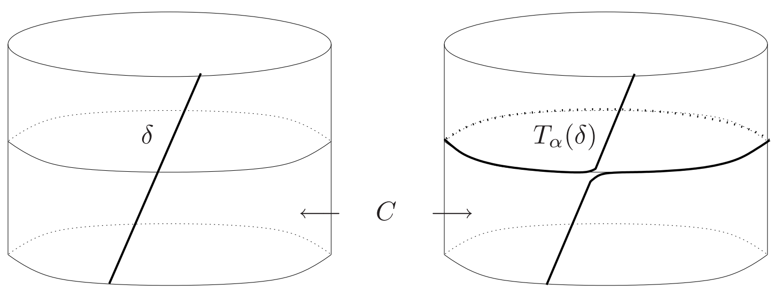

FIGURE 1. A representative of the image of an arc δ under T_α

Because α_q and β_q are transverse, the number $\mathrm{i}(\alpha,\beta)$ counts the number of intersection points of α_q and β_q which in turn counts the number of arcs δ of intersection that β_q makes with C. The length of the representative $T_\alpha(\beta)$ we have constructed is thus precisely

$$\ell_q(\beta) + \ell_q(\alpha) \cdot \mathrm{i}(\alpha,\beta).$$

As we noted above, our representative is not geodesic, and hence

$$\ell_q(T_\alpha(\beta)) < \ell_q(\beta) + \ell_q(\alpha) \cdot \mathrm{i}(\alpha,\beta).$$

Therefore (1) is not satisfied, proving the first half of the lemma, since β was arbitrary.

We now assume $\alpha \notin \mathsf{cyl}(q)$, and find β with $\mathrm{i}(\alpha,\beta) \neq 0$ so that (1) is satisfied. Assume for simplicity that S is closed (the punctured case is similar). Consider the universal cover $\widetilde{S}$ of S equipped with the lifted metric of q, and fix a lift $\widetilde{\alpha}_q$ of α_q. The biinfinite geodesic $\widetilde{\alpha}$ separates $\widetilde{S}$ into two components, $H^+ \cup H^-$. Let h be an element of $\pi_1(S)$ that generates the stabilizer of $\widetilde{\alpha}_q$, so that its action is by translation along $\widetilde{\alpha}_q$.

Because α is not a cylinder curve, $\widetilde{\alpha}_q$ is a concatenation of saddle connections meeting at singularities of $\widetilde{q}$. Consider the angles made on each of the two sides at the singularities. If the angles were always π on one side, then there is a parallel curve on S that is nonsingular, which means α itself is in $\mathsf{cyl}(q)$, contrary to assumption. Thus, there is a singularity x^+ so that the angle at x^+ on the H^+

side made by the saddle connections meeting there is strictly greater than π, and likewise there is x^- chosen relative to H^-.

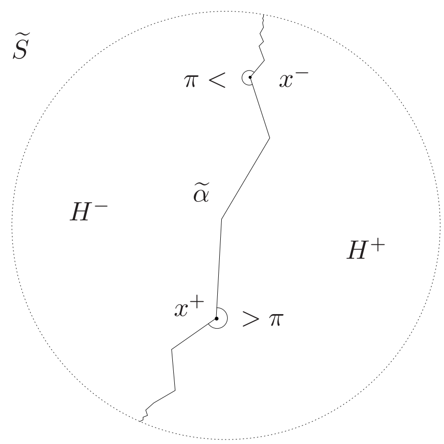

FIGURE 2. The lift $\widetilde{\alpha}$ and the singularities $x^{\pm}$.

There are geodesics $\gamma^{\pm}$ contained in $H^{\pm}$ meeting $\widetilde{\alpha}_q$ precisely in the points $x^{\pm}$. Let A^+ (respectively A^-) be the region on the circle at infinity bounded by an endpoint of γ^+ (respectively γ^-) and an endpoint of $\widetilde{\alpha}_q$ as shown in Figure 3.

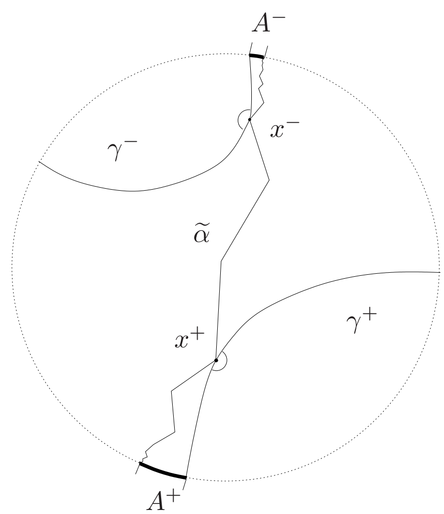

FIGURE 3. The intervals along the boundary.

Let $\beta_0 \in \mathcal{S}$ be any curve with $\mathrm{i}(\alpha, \beta_0) = k \neq 0$. For each $1 \leq j \leq k$, we pick a lift $\widetilde{\beta}^j$ whose endpoints link those of $\widetilde{\alpha}$. By replacing β_0 with its image $\beta = T_\alpha^N(\beta_0)$ for large enough N, we can choose $\widetilde{\beta}^j$ so that it has one endpoint in A^+ and the other in A^-, since the effect of T_α is to shear along $\widetilde{\alpha}$. Observe that each such $\widetilde{\beta}^j$ includes the $\tilde{q}$–geodesic segment $[x^+, x^-]$. Therefore, for each of the k essential intersections of β with α, the curve β traverses some definite length of α. It follows that the geodesic representative of $T_\alpha(\beta)$ is now exactly obtained from β by surgering in k copies of α traversed. From this, we get (1), as required. □

Corollary 18. *If $q, q' \in \mathsf{Flat}(S)$ and $\lambda_{\mathcal{S}}(q) = \lambda_{\mathcal{S}}(q')$, then $\mathsf{cyl}(q) = \mathsf{cyl}(q')$.* □

The next lemma, combined with Lemma 17, reduces the subspace of $\mathsf{Flat}(S)$ having prescribed lengths to a disk.

Lemma 19. *If $\mathsf{cyl}(q) = \mathsf{cyl}(q')$, then $\mathbb{H}_q = \mathbb{H}_{q'}$.*

Proof. Suppose $\mathsf{cyl}(q) = \mathsf{cyl}(q')$. First lift q and q' to arbitrary representatives in $\mathcal{Q}^1$, also called q and q', so that it is well-defined to talk about particular directions. Note that a cylinder curve, since it has a parallel family of nonsingular representatives, has a well-defined direction $\theta \in \mathbb{RP}^1$. Next, recall that for any quadratic differential, the set of directions with at least one cylinder is dense in $\mathbb{RP}^1$ by a result of Masur [21]. Thus, for every uniquely ergodic foliation $\nu_q^\theta \in \mathcal{PMF}(q)$, there is a sequence of cylinder curves $\alpha_i \in \mathsf{cyl}(q)$ for which the directions converge: $\theta_i \to \theta$. It follows that

$$\nu_q^{\theta_i} \to \nu_q^\theta \qquad \text{as} \qquad i \to \infty.$$

Since $\mathrm{i}(\nu_q^{\theta_i}, \alpha_i) = 0$, it follows that in $\mathcal{PMF}$, up to subsequence, we have $\alpha_i \to \mu \in \mathcal{PMF}$ with $\mathrm{i}(\mu, \nu_q^\theta) = 0$. Since ν_q^θ is uniquely ergodic, this means that $\mu = \nu_q^\theta \in \mathcal{PMF}$, and hence $\alpha_i \to \nu_q^\theta$ in $\mathcal{PMF}$. From the assumption that $\mathsf{cyl}(q') = \mathsf{cyl}(q)$, it follows that ν is also in $\mathcal{PMF}(q')$. Thus the sets of uniquely ergodic foliations in $\mathcal{PMF}(q)$ and $\mathcal{PMF}(q')$ are identical.

Consider a pair of uniquely ergodic foliations μ_0 and ν_0 in $\mathcal{PMF}(q) \cap \mathcal{PMF}(q')$. There is a matrix M (respectively, M') in $SL_2(\mathbb{R})$ so that μ_0 and ν_0 are the vertical and the horizontal foliations of Mq (respectively, $M'q'$). However, there is a unique Teichmüller geodesic connecting μ_0 and ν_0 ([13]). Therefore,

$$M'q' = A_t M q \qquad \text{for} \qquad A_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

That is, q' is in the $SL(2,\mathbb{R})$ orbit of q and hence $\mathbb{H}_q = \mathbb{H}_{q'}$. □

Proof of Theorem 1. Suppose $\lambda_{\mathcal{S}}(q) = \lambda_{\mathcal{S}}(q')$. By Lemma 17, $\mathsf{cyl}(q) = \mathsf{cyl}(q')$ and so Lemma 19 implies $\mathbb{H}_q = \mathbb{H}_{q'}$. A level set of the length of a given cylinder curve on $\mathbb{H}_q = \mathbb{H}_{q'}$ is a horocycle. So if $\alpha, \beta, \gamma \in \mathsf{cyl}(q) = \mathsf{cyl}(q')$ have distinct directions, then q and q' are contained in the intersection of the same three distinct horocycles. As in the case of flat tori (Proposition 16), this implies $q = q'$. □

4. Iso-length-spectral families

Here we show constructively that for a set of curves to be spectrally rigid, its projectivization must not miss any open set of $\mathcal{PMF}$.

Theorem 3. *Suppose $\xi(S) \geq 2$. If $\Sigma \subset \mathcal{S} \subset \mathcal{PMF}$ and $\overline{\Sigma} \neq \mathcal{PMF}$, then there is a deformation family $\Omega_\Sigma \subset \mathsf{Flat}(S)$ for which $\Omega_\Sigma \to \mathbb{R}^\Sigma$ is constant, and such that the dimension of Ω_Σ is proportional to the dimension of $\mathsf{Flat}(S)$ itself.*

In particular, no finite set of curves determines a flat metric. We will build deformation families of flat metrics in this section based on a train track argument. We refer the reader to [26] for a detailed discussion of train tracks.

Given a metric ρ on S (with metric completion $\hat{S}$), we call a train track $\tau \subset S$ *magnetic* with respect to ρ if there exists a map $f : (\hat{S}, P) \to (\hat{S}, P)$, homotopic to the identity rel P, such that if $\gamma \subset \tau$ is a curve carried by τ, then $f(\gamma)$ is a ρ–geodesic representative of γ (up to parametrization). The magnetizing map f should be thought of as taking a smooth realization of the train track to a geodesic realization (compare Figure 7 below). In the examples in this section, f is a homeomorphism isotopic to the identity. More complicated maps f are used to deal with the case of punctures, and are presented in the appendix.

Informally, a train track is magnetic if geodesics "stick to it": geodesics carried by τ actually live inside of the one-complex $f(\tau)$ as concatenations of the branches. Note that while magnetic train tracks are easily constructed for flat metrics, they do not exist for any hyperbolic metric (or in fact for any complete Riemannian metric).

The strategy for proving Theorem 3 is to first construct an initial train track τ on S and a deformation family $\Omega \subset \mathsf{Flat}(S)$ so that τ is magnetic in q for all $q \in \Omega$ and so that the length of any curve γ carried by τ is constant on Ω. The train track τ we construct is complete and recurrent, hence the subset $U_\tau \subset \mathcal{PML}$ consisting of laminations carried by τ has nonempty interior. Then, if $\Sigma \subset \mathcal{S}$ is not dense, we will find a mapping class ψ adapted to Σ such that $\Sigma \in \psi U_\tau = U_{\psi\tau}$, and the deformation family promised in the theorem will then be $\psi\Omega$.

The main ingredient needed to prove Theorem 3 is thus the following.

Proposition 20. *If $\xi(S) \geq 2$, then there exists a complete recurrent train track τ and a positive-dimensional family of flat structures $\Omega \subset \mathsf{Flat}(S)$ such that:*

- *τ is magnetic in q for all $q \in \Omega$; and*
- *the length of any curve γ carried by τ is constant on Ω.*

Proof. If τ is a magnetic train track for ρ, then there is a nonnegative length vector assigned to each branch of $f(\tau)$. The ρ–length of any curve carried by τ can be computed as the dot product of the weight vector for the curve with the length vector, and the allowable weight vectors are precisely those meeting the switch conditions. Thus, we must construct the family Ω so that the *difference* between the length vectors for any two $q, q' \in \Omega$ lies in the orthogonal complement of the space of weight vectors on τ. Geometrically, this means that the difference in length vectors for $q, q' \in \Omega$ can be distributed among the switches so that at each switch, the increase in length of the incoming branches is exactly equal to the decrease in length for each outgoing branches; see Figure 4.

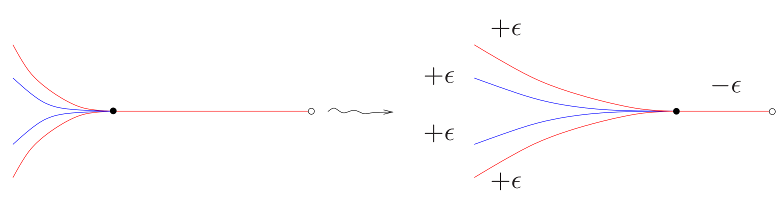

FIGURE 4. Changing the length vectors can be thought of as folding or unfolding at switches, and leaves invariant the length of curves carried by the train track.

The idea is to build families of metrics and train tracks on basic building blocks, then glue them together to obtain S. For simplicity, we only provide the details for closed surfaces in this section, as these can all simultaneously be handled by constructing a single building block. To prove the theorem for all surfaces S with

$\xi(S) \geq 2$ it suffices to construct six more building blocks, using the same general ideas. For completeness, we have included a description of these remaining building blocks in an appendix at the end of the paper.

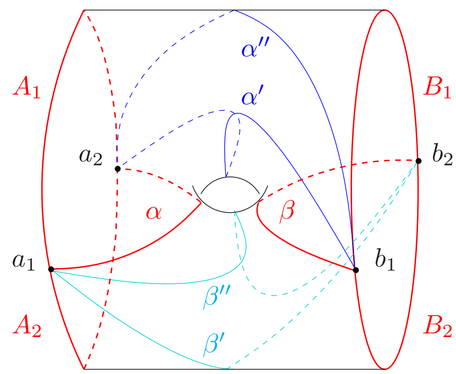

FIGURE 5. One basic building block Δ and its train track τ. The cylinder C_1 is pictured on the top and C_2 on the bottom. Copies of Δ can be glued together end-to-end to obtain a copy of S.

The basic building block Δ is a genus-one surface with two boundary components described here and shown in Figure 5. We will put a metric and a train track on Δ, and then assemble S from $g-1$ copies of Δ by gluing the boundary components in pairs. Choose nonperipheral arcs α (with endpoints a_1, a_2) and β (endpoints b_1, b_2) joining each boundary component to itself. Then the complement of those arcs is a pair of annuli. For any choice of $t > 0$, there is a unique flat metric on Δ so that $\ell(\alpha) = \ell(\beta) = t$, and the two complementary annuli C_i are Euclidean cylinders with boundary lengths $2t$ and heights t (shown in Figure 6). This means each cylinder will have area $2t^2$, so Δ will have area $4t^2$.

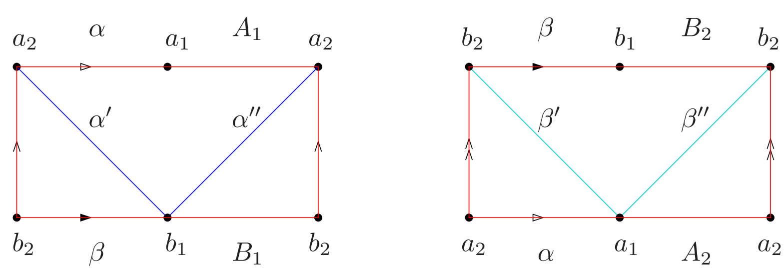

FIGURE 6. Metric pictures of the two cylinders C_1 (left) and C_2 (right) which make up Δ.

Choose the value of t so that $4t^2(g-1) = 1$ (in order that the glued surface will have total area one). After gluing $g-1$ copies of Δ together end to end, we obtain a flat metric q_0 on S, whose singular points come from the a_i and b_i in the pieces Δ. We will choose to initially glue with a quarter-twist (compare Figure 9), so that there are four evenly spaced vertices around the gluing curves, and the singularities all have cone angle 3π.

Next we build a one-complex T_0 of geodesic segments in q_0. In each piece Δ, let α', α'' be the minimal-length segments connecting a_2 to b_1 in C_1, and likewise β', β'' connecting a_1 to b_2 in C_2 (the length of each of these will be $\sqrt{2}t$); see Figure 5. Then the branches of T_0 are the saddle connections which belong to the boundary of a piece Δ, together with the arcs $\alpha, \alpha', \alpha'', \beta, \beta', \beta''$ in those pieces. There are switches for T_0 at all of the singularities in the flat metric q_0.

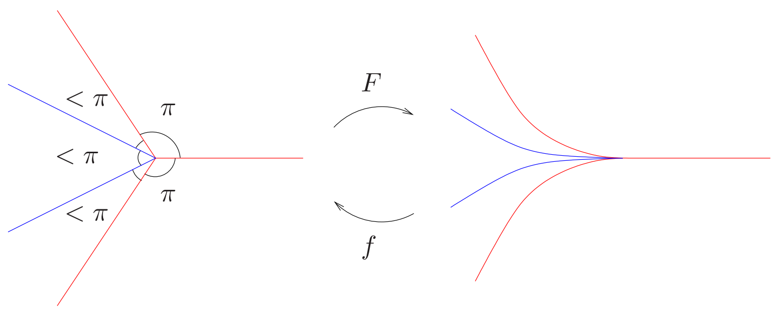

FIGURE 7. The homeomorphisms of S pictured here map between a geodesic one-complex T and a train track τ. This figure shows how to use the angles in T to read off the illegal turns at each switch, which specifies the tangent spaces for τ. The inverse map f is the magnetizing homeomorphism for τ with respect to the flat metric.

Each 1–cell of this complex T_0 is smoothly embedded in S. However, there is no well-defined tangent space at the switches. To obtain a train track τ, we apply an appropriate homeomorphism F which is isotopic to the identity. That is, we must specify at each switch which branches are incoming and which are outgoing. For all of the complexes T in the deformation family, every switch in T will have total angle 3π and five incident branches, one of which is separated from its neighboring branches by angle π on each side. This determines the tangencies as in Figure 7.

Any curve $\gamma \subset \tau$ is mapped by $f = F^{-1}$ to a concatenation of geodesic segments which are branches of $T_0 = f(\tau)$. But then they meet the angle conditions that suffice for geodesity (Remark 8), so τ is magnetic with respect to q_0. The complementary regions are triangles, so τ is complete, and it is straightforward to construct a positive measure on τ, thus showing that it is recurrent.

Next we describe a deformation space Ω of q_0 so that a choice of parameters specifies a modified 1–complex T (combinatorially equivalent to T_0 but with new lengths prescribed by the parameters) and a modified flat metric q, so that the lengths of curves carried by τ do not change as the parameters vary. This will establish that τ remains magnetic in q over the whole family Ω. The deformations can be carried out independently in each block, provided we keep track of the gluing information. In each Δ, the deformations will be parameterized by two numbers ϵ and δ (small compared to t) as follows.

Each new metric is built from Euclidean cylinders glued along arcs in the boundary with the same combinatorial pattern as that of q_0. There are four switches and 10 arcs in Δ. To guarantee that the resulting one-complex is realizable as a train track, it is necessary that the switch conditions be preserved; this is equivalent to

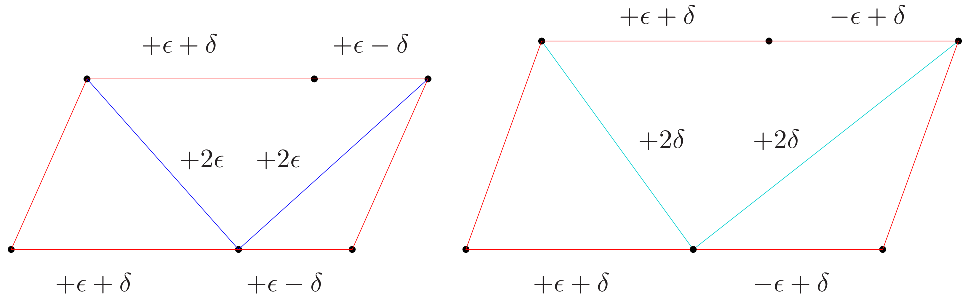

FIGURE 8. We have two parameters ϵ, δ to perturb the flat structures in each piece Δ. Metrically, this can be achieved by deforming the rectangles to parallelograms, adjusting the height and shear appropriately. (Compare Figure 6.)

requiring that the perturbations be in the row space of the following matrix.

	A_1	A_2	B_1	B_2	α	α'	α''	β	β'	β''
a_1	-1	$+1$	0	0	$+1$	0	0	0	$+1$	$+1$
a_2	$+1$	-1	0	0	$+1$	$+1$	$+1$	0	0	0
b_1	0	0	$+1$	-1	0	$+1$	$+1$	$+1$	0	0
b_2	0	0	-1	$+1$	0	0	0	$+1$	$+1$	$+1$

A priori, this gives four degrees of freedom. However, in order for the metric cylinder picture to be preserved, we further require two geometric conditions on the lengths of the curves:

$$A_1 + \alpha = B_1 + \beta \qquad \text{and} \qquad A_2 + \alpha = B_2 + \beta,$$

which say that the top and bottom circumferences are equal for each of C_1 and C_2. In fact this is necessary and sufficient for the realization by metric Euclidean cylinders, as depicted in Figure 8. (Note that the boundary components of Δ automatically have equal length because $A_1 + A_2 = B_1 + B_2$ is satisfied for any perturbation satisfying the switch conditions.)

It follows that there are two free parameters, which we can record according to the table below.

A_1	A_2	B_1	B_2	α	α'	α''	β	β'	β''
$-\epsilon+\delta$	$+\epsilon-\delta$	$-\epsilon+\delta$	$+\epsilon-\delta$	$+\epsilon+\delta$	$+2\epsilon$	$+2\epsilon$	$+\epsilon+\delta$	$+2\delta$	$+2\delta$

And indeed the gluing of neighboring pieces Δ_i is also prescribed by the same parameters, as illustrated in Figure 9.

It is immediate, by construction, that the lengths of curves $\gamma \subset \tau$ are preserved as these parameters vary, since changes to the length are compensated at every switch.

If we write $(\bar{\epsilon}, \bar{\delta}) = (\epsilon_1, \delta_1, \ldots, \epsilon_{g-1}, \delta_{g-1})$ for the vector of the parameters, then we obtain a $2(g-1)$–dimensional deformation space from the perturbed metrics $\{q_0(\bar{\epsilon}, \bar{\delta})\}$. We let

$$\Omega = \{q_0(\bar{\epsilon}, \bar{\delta})\} \cap \mathsf{Flat}(S),$$

which is the subspace with unit area; this has codimension 1, so $\dim(\Omega) = 2g-3$. □

We can now prove Theorem 3 by finding a mapping class to apply to Σ so that all of the image curves are carried by τ.

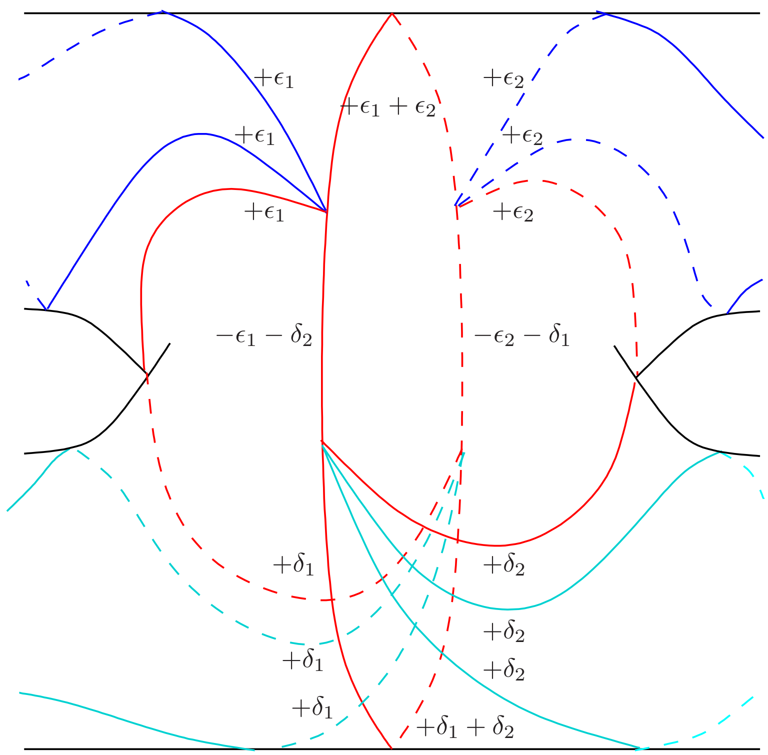

FIGURE 9. The changes in lengths assigned to τ near one of the gluing curves. On the left side the deformations are parameterized by (ϵ_1, δ_1) in the block Δ_1 and on the right by (ϵ_2, δ_2) in the block Δ_2.

Proof of Theorem 3. Let $\Omega \subset \mathsf{Flat}(S)$ and $\tau \subset S$ be as in Proposition 20. Since τ is complete and recurrent, the subset $U_\tau \subset \mathcal{PML}$ consisting of those measured laminations carried by τ has nonempty interior. Let $h \in \mathrm{Mod}(S)$ be a pseudo-Anosov mapping class whose attracting point in $\mathcal{PML}$ is a lamination $\lambda^+ \in U_\tau$. By assumption, Σ is not dense, so there is an open set $W \in \mathcal{PML}$ such that $\Sigma \cap W = \emptyset$.

Since any orbit of the mapping class group is dense in $\mathcal{PML}$, there is some mapping class $\varphi \in \mathrm{Mod}(S)$ such that $\lambda^- \in \varphi W$, where λ^- is the repelling lamination of h. But then $\varphi\Sigma$ misses a neighborhood of λ^-, so for n sufficiently large, any curve in $h^n\varphi\Sigma$ is carried by τ. Equivalently, any curve in Σ is carried by $\varphi^{-1}h^{-n}\tau$.

Now we set

$$\Omega_\Sigma = \{\varphi^{-1}h^{-n}q \,|\, q \in \Omega\},$$

and observe that the length of any curve $\gamma \in \Sigma$ is constant on Ω_Σ since it is carried by $\varphi^{-1}h^{-n}(\tau)$, and the property of being magnetic is clearly preserved when both the train track and the metric are modified by the same mapping class. $\square$

Remark 21. Here, we obtain a deformation family of dimension $2g-3$. We make no claim that this is optimal, but note that the optimal dimension is bounded above and below by linear functions in g, since $\mathsf{Flat}(S)$ itself has dimension $12g-$

14. For the cases covered in the appendix, which allow punctures and boundary components, this proportionality holds as well: the the number of parameters in the deformation space is linearly comparable to $g + n + b$, as is the complexity of S and therefore the dimension of $\mathsf{Flat}(S)$.

5. Flat structures as currents

Bonahon's space of geodesic currents derives its utility from the fact that so many spaces embed into it in natural ways with respect to the intersection form. For example, the space of measured laminations $\mathcal{ML}$, being the completion of $\mathcal{S}$ with respect to i, is easily seen to embed into $\mathsf{C}(S)$, and the restriction of i to $\mathcal{ML} \times \mathcal{ML}$ is Thurston's continuous extension of geometric intersection number from weighted simple curves to measured laminations. In this section, we see that $\mathsf{Flat}(S)$ embeds naturally as well.

For closed surfaces, Bonahon constructs an embedding of $\mathcal{T}(S)$ into $\mathsf{C}(S)$ in [5] by sending a hyperbolic metric m to its associated Liouville current L_m. This was extended to all negatively curved Riemannian metrics by Otal in [24] and to negatively curved cone metrics by Hersonsky–Paulin in [16]. Given any such metric m, we will denote the associated current by L_m. The naturality with respect to i is expressed by the equation

$$\mathrm{i}(L_m, \alpha) = \ell_m(\alpha).$$

This extends easily to $\mathsf{Flat}(S)$, and in fact it is possible to carry out this construction for surfaces which are not necessarily closed. Given $q \in \mathcal{Q}^1(S)$, we can view $\theta \mapsto \nu_q^\theta$ as a map $\mathbb{RP}^1 \to \mathsf{C}(S)$.

Proposition 22. *For any $q \in \mathsf{Flat}(S)$ there exists a current L_q such that*

(1) *for all $\alpha \in \mathcal{C}'$,* $\quad \mathrm{i}(L_q, \alpha) = \ell_q(\alpha)$*;*

(2) *for all $\mu \in \mathsf{C}(S)$ and any $q \in \mathcal{Q}^1(S)$ inducing the given $q \in \mathsf{Flat}(S)$,*

$$\mathrm{i}(L_q, \mu) = \frac{1}{2}\int_0^\pi \mathrm{i}(\nu_q^\theta, \mu)\, d\theta;$$

(3) $\mathrm{i}(L_q, L_q) = \pi/2$.

Proof. We can define L_q by a Riemann integral

$$L_q = \frac{1}{2}\int_0^\pi \nu_q^\theta d\theta$$

by which we mean a limit of Riemann sums. Since $\mathbb{RP}^1$ is compact, the map $f(\theta) = \nu_q^\theta$ is uniformly continuous. As d is complete, this integral exists.

For any $\alpha \in \mathcal{C}'$, we recall the formula from Lemma 7

$$\ell_q(\alpha) = \frac{1}{2}\int_0^\pi \mathrm{i}(\nu_q^\theta, \alpha) d\theta.$$

Combining this with the uniform continuity of ν_q^θ implies part (1) and also part (2) for any current μ which is a scalar multiple of a current associated to a curve. For general currents we appeal to the density of $\mathbb{R}_+ \times \mathcal{C}$ in $\mathsf{C}(S)$ and the continuity of intersection number. Since the foliations ν_q^θ have q-length 1 (and so $\mathrm{i}(L_q, \nu_q^\theta) = 1$), the third statement follows from the second statement by the computation

$$\mathrm{i}(L_q, L_q) = \frac{1}{2}\int_0^\pi \mathrm{i}(L_q, \nu_q^\theta)\, d\theta = \frac{1}{2}\int_0^\pi d\theta = \frac{\pi}{2}. \qquad \square$$

Remark 23. In the closed case, an equivalent definition of L_q can be given as a cross-ratio, as in Hersonsky-Paulin.

The embedding of $\mathsf{Flat}(S)$ in $\mathsf{C}(S)$ is now immediate.

Theorem 4. *There is an embedding*

$$\mathsf{Flat}(S) \to \mathsf{C}(S)$$

denoted by $q \mapsto L_q$ *so that for* $q \in \mathsf{Flat}(S)$ *and* $\alpha \in \mathcal{C}'$*, we have* $\mathrm{i}(L_q, \alpha) = \ell_q(\alpha)$*. Furthermore, after projectivizing,* $\mathsf{Flat}(S) \to \mathcal{P}\mathsf{C}(S)$ *is still an embedding.*

Proof. If $q_n \to q$ in $\mathsf{Flat}(S)$, then $\ell_{q_n}(\alpha) \to \ell_q(\alpha)$, and hence $L_{q_n} \to L_q$ by Theorems 11 and 15. Thus, $q \mapsto L_q$ is continuous.

Injectivity for $\mathsf{Flat}(S) \to \mathsf{C}(S)$ follows directly from Theorem 1, where it is shown that even intersection with elements of $\mathcal{S}$ distinguishes flat metrics. Injectivity for $\mathsf{Flat}(S) \to \mathcal{P}\mathsf{C}(S)$ follows from the fact that $\mathrm{i}(L_q, L_q)$ is constant, which ensures that no two currents in the image of $\mathsf{Flat}(S)$ can be multiples of one another.

Finally, to see that these maps are embeddings, we need only show that if q_n exits every compact set in $\mathsf{Flat}(S)$, then L_{q_n} has no subsequence which converges to a point of (the image of) $\mathsf{Flat}(S)$. To see this, observe that if the lengths of all simple closed curves were bounded away from zero and infinity as $n \to \infty$, then q_n would stay in a compact part of $\mathsf{C}(S)$. So first suppose there exists $\gamma \in \mathcal{S}$ for which

$$\mathrm{i}(L_{q_n}, \gamma) = \ell_{q_n}(\gamma) \to \infty.$$

In this case $L_{q_n} \to \infty$ in $\mathsf{C}(S)$ and (as we show in the proof of Theorem 6) any projectively convergent subsequence L_{q_n} must converge to a measured lamination, thus exiting $\mathsf{Flat}(S)$. The second possibility is that there is a lamination $\lambda \in \mathcal{ML}$ with

$$\mathrm{i}(L_{q_n}, \lambda) = \ell_{q_n}(\lambda) \to 0.$$

Since $\mathrm{i}(L_q, \lambda) > 0$ for any $q \in \mathsf{Flat}(S)$, it follows that any limit of L_{q_n} does not lie in $\mathsf{Flat}(S)$. $\square$

As a consequence of the embedding, we find that the length of a lamination in a flat metric is well-defined.

Corollary 24. *The flat-length function* $\mathsf{Flat}(S) \times \mathcal{S}(S) \to \mathbb{R}$ *has a continuous homogeneous extension*

$$\overline{\ell} : \mathsf{Flat}(S) \times \mathcal{MF}(S) \to \mathbb{R}.$$

given by

$$(q, \mu) \mapsto \overline{\ell}_q(\mu) = \mathrm{i}(L_q, \mu).$$

We can now prove the main theorem.

Theorem 2. *If* $\xi(S) \geq 2$*, then* $\Sigma \subset \mathcal{S} \subset \mathcal{PMF}$ *is spectrally rigid over* $\mathsf{Flat}(S)$ *if and only if* Σ *is dense in* $\mathcal{PMF}$*.*

Proof. We first assume Σ is dense in $\mathcal{PMF}$. Suppose $q, q' \in \mathsf{Flat}(S)$ have $\ell_q(\alpha) = \ell_{q'}(\alpha)$ for all $\alpha \in \Sigma$. For any $\mu \in \mathcal{MF}$, the density hypothesis implies that there are scalars t_i and curves $\alpha_i \in \Sigma$ such that $t_i\alpha_i \to \mu$. But

$$\ell_q(t_i\alpha_i) = \ell_{q'}(t_i\alpha_i),$$

so Corollary 24 implies ℓ_q and $\ell_{q'}$ agree on μ. In particular, the two metrics assign the same length to all simple closed curves. By Theorem 1, it follows that $q = q'$, and thus Σ is spectrally rigid.

Next assume that Σ is not dense in $\mathcal{PMF}$. Theorem 3 implies the existence of a positive-dimensional family $\Omega_\Sigma \subset \mathsf{Flat}(S)$ for which the lengths of curves in Σ is constant. In particular, there exists a pair of distinct flat structures $q, q' \in \Omega_\Sigma$ for which $\ell_q(\alpha) = \ell_{q'}(\alpha)$ for all $\alpha \in \Sigma$, and hence Σ is not spectrally rigid. □

6. The boundary of $\mathsf{Flat}(S)$

In this section we give a description of the geodesic currents that appear in the closure of $\mathsf{Flat}(S) \subset \mathcal{P}\mathsf{C}(S)$. We will show that the limit points have geometric interpretations as a hybrid of a flat structure on some subsurface and a geodesic lamination on a disjoint subsurface (Theorem 6). We call such currents *mixed structures*. As a first step, we show that the description of L_q as average intersection number with foliations ν_q^θ (Proposition 22, part (2)) extends to any limiting geodesic current. This description greatly simplifies the analysis of what geodesic currents can appear as degenerations of flat metrics.

To every nonzero quadratic differential, we consider again the map

$$\mathbb{RP}^1 \to \mathcal{ML}(q) \subset \mathcal{ML}(S)$$

sending $\theta \mapsto \nu_q^\theta$, the foliation in direction θ. We show that given a sequence of quadratic differentials whose associated currents converge in $\mathsf{C}(S)$, the maps $\theta \to \nu_q^\theta$ converge uniformly (up to subsequence) to a continuous map from $\mathbb{RP}^1$ to $\mathcal{ML}(S)$.

Lemma 25. *For every $q \in \mathcal{Q}^1(S)$, $\alpha \in \mathcal{C}'$, and angles θ_0 and θ_1, we have*

$$\left| \mathrm{i}(\nu_q^{\theta_1}, \alpha) - \mathrm{i}(\nu_q^{\theta_0}, \alpha) \right| \le \ell_q(\alpha)|\theta_1 - \theta_0|.$$

It follows that $\theta \mapsto \nu_q^\theta$ is Lipschitz.

Proof. Let ω be a saddle connection contained in a q–geodesic representative of α. Assume ω has an angle ϕ. We have $\mathrm{i}(\nu_q^\theta, \omega) = \ell_q(\omega)|\sin(\theta - \phi)|$. Hence

$$\left| \frac{d}{d\theta}\,\mathrm{i}(\nu_q^\theta, \omega) \right| = \ell_q(\omega)|\cos(\theta - \phi)| \le \ell_q(\omega).$$

Integrating the above inequality from θ_0 to θ_1 and adding up over all saddle connections of α proves the lemma. □

Proposition 26. *Let q_n be a sequence of quadratic differentials so that $s_n L_{q_n}$ converges in $\mathsf{C}(S)$ to a geodesic current L_∞. Then, after possibly passing to a subsequence, the sequence of functions*

$$f_n \colon \mathbb{RP}^1 \to \mathcal{ML}(S), \qquad f_n(\theta) = s_n \nu_{q_n}^\theta$$

converges uniformly to a continuous function

$$f_\infty \colon \mathbb{RP}^1 \to \mathcal{ML}(S).$$

Proof. We can consider f_n as maps from $\mathbb{RP}^1$ to $\mathsf{C}(S)$. Since $\mathcal{ML}(S)$ is a closed subset of $\mathsf{C}(S)$, the image of the limiting map f_∞, provided it exists, is automatically in $\mathcal{ML}(S)$.

Equip $\mathsf{C}(S)$ with the metric in Theorem 11 (or 15 for punctured surfaces). By the Arzelá-Ascoli theorem, it is sufficient to show that the family of maps f_n are equicontinuous with respect to this metric and the union of the images have compact closure. For angles θ_0 and θ_1 we have

$$\begin{aligned} d\big(f_n(\theta_1), f_n(\theta_0)\big) &= \sum_{\alpha \in \mathcal{C}'(S)} s_n t_\alpha \Big| \mathrm{i}(\nu_{q_n}^{\theta_1}, \alpha) - \mathrm{i}(\nu_{q_n}^{\theta_0}, \alpha) \Big| \\ &\leq |\theta_1 - \theta_0| \sum_{\alpha \in \mathcal{C}'(S)} s_n t_\alpha \, \ell_{q_n}(\alpha) \\ &= |\theta_1 - \theta_0| \cdot d(s_n L_{q_n}, 0). \end{aligned}$$

The inequality follows from Lemma 25, and the equalities are immediate from the definition the metric, together with Proposition 22. Since

$$d(s_n L_{q_n}, 0) \to d(L_\infty, 0),$$

there exists $K > 0$ such that $d(s_n L_{q_n}, 0) \leq K$, and so the family of maps $\{f_n\}$ is equicontinuous.

It remains to show that the $\cup_n f_n(\mathbb{RP}^1)$ has compact closure. Observe that

$$\mathrm{i}(f_n(\theta), \alpha) = \mathrm{i}(s_n \nu_q^\theta, \alpha) \leq s_n \ell_{q_n}(\alpha).$$

Therefore,

$$d(f_n(\theta), 0) \leq \sum_{\alpha \in \mathcal{C}'(S)} t_\alpha \ \mathrm{i}(f_n(\theta), \alpha) \leq \sum_{\alpha \in \mathcal{C}'(S)} s_n t_\alpha \ell_{q_n}(\alpha) = d(s_n L_{q_n}, 0).$$

and so $\cup_n f_n(\mathbb{RP}^1)$ is contained in the closed K–ball about 0. Since d is proper, this ball is compact. □

We now define *mixed structures* on S. This requires us to first make precise what we will mean by a flat structure on a subsurface.

Suppose $X \subset S$ is a π_1–injective subsurface of S with $\chi(X) < 0$. We view X as a punctured surface (removing every boundary component), and let $\mathsf{Flat}(X)$ denote the space of flat structures on X. By this we mean a flat structure on each component of X as described in Section 2.3, where we now require the sum of the areas of the components to be one. Observe that the boundary curves of X are realized by punctures and hence have length 0. Equivalently, an element of $\mathsf{Flat}(X)$ is given by a unit norm quadratic differential in $\mathcal{Q}(X)$, nonzero on all components, and well-defined up to multiplication by a unit norm complex number in each component. Representing any $q \in \mathsf{Flat}(X)$ by a unit norm quadratic differential, we have the map $\mathbb{RP}^1 \to \mathcal{ML}(X)$ given by $\theta \mapsto \nu_q^\theta$ as before. Extending measured laminations on X to measured laminations on S in the usual way, we can view $\theta \mapsto \nu_q^\theta$ as a map into $\mathcal{ML}(S) \subset \mathsf{C}(S)$.

Given a subsurface $X \subset S$ as above, $q \in \mathsf{Flat}(S)$, and a measured lamination $\lambda \in \mathcal{ML}(S)$ whose support can be homotoped disjoint from X, we define a mixed structure $\eta = (X, q, \lambda)$ to be the geodesic current given by

$$\eta = \lambda + \frac{1}{2} \int_0^\pi \nu_q^\theta d\theta.$$

Here the integral is a Riemann integral, as in the proof of Proposition 22. We sometimes write $\eta = \lambda + L_q$. It follows that for every $\alpha \in \mathcal{C}'(S)$,

$$\mathrm{i}(\eta, \alpha) = \mathrm{i}(\lambda, \alpha) + \frac{1}{2}\int_0^\pi \mathrm{i}(\nu_q^\theta, \alpha)\, d\theta.$$

We also allow the two degenerate situations $X = S$ and $X = \emptyset$. In these cases, the corresponding mixed structure is a flat structure on S or a measured lamination on S, respectively.

Now let $\mathsf{Mix}(S) \subset \mathsf{C}(S)$ denote the space of all mixed structures, and $\mathcal{P}\mathsf{Mix}(S)$ its image in $\mathcal{P}\mathsf{C}(S)$ under the projection $\mathsf{C}(S) \to \mathcal{P}\mathsf{C}(S)$. Observe that if

$$\eta \in \mathsf{Mix}(S) \setminus \mathcal{ML}(S)$$

then $\mathrm{i}(\eta, \eta) = \pi/2$, just as in Proposition 22.

Note that if α is a curve in ∂X, then $\mathrm{i}(\nu_q^\theta, \alpha) = 0$ and $\mathrm{i}(\lambda, \alpha) = 0$. Hence $\mathrm{i}(\eta, \alpha) = 0$, although α may be contained in the support of λ (and thus η).

Theorem 6. *The closure of* $\mathsf{Flat}(S)$ *in* $\mathcal{P}\mathsf{C}(S)$ *is exactly the space* $\mathcal{P}\mathsf{Mix}(S)$*. That is, for any sequence* $\{q_n\}$ *in* $\mathsf{Flat}(S)$*, after passing to a subsequence if necessary, there exists a mixed structure* η *and a sequence of positive real numbers* $\{t_n\}$ *so that*

$$\lim_{n\to\infty} t_n \ell_{q_n}(\alpha) = \mathrm{i}(\alpha, \eta).$$

for every $\alpha \in \mathcal{C}$*. Moreover, every mixed structure is a limit of a sequence in* $\mathsf{Flat}(S)$*.*

Proof. Let q_n be a sequence of quadratic differentials such that $t_n L_{q_n} \to L_\infty$, for positive real numbers t_n. We have to show that up to scaling, $L_\infty \in \mathsf{Mix}(S)$.

If the sequence t_n converges to zero then

$$\mathrm{i}(L_\infty, L_\infty) = \lim_{n\to\infty} t_n^2\, \mathrm{i}(L_{q_n}, L_{q_n}) = \frac{\pi}{2} \lim_{n\to\infty} t_n^2 = 0.$$

That is, L_∞ is a measured lamination (c.f. Bonahon [5]). Thus the theorem holds with $X = \emptyset$.

Since every geodesic current has finite self-intersection number, we can conclude that t_n does not tend to infinity. Therefore, after taking a subsequence, we can assume that the sequence t_n is convergent, and in fact converges to 1. That is, there is a geodesic current (which we again denote by L_∞) such that $L_{q_n} \to L_\infty$ in $\mathsf{C}(S)$. Applying Proposition 26 and taking a further subsequence if necessary, we can also assume that f_n converges uniformly to a continuous map f_∞. As a consequence, for every curve $\alpha \in \mathcal{C}$,

$$\mathrm{i}(L_\infty, \alpha) = \frac{1}{2}\int_0^\pi \mathrm{i}\big(f_\infty(\theta), \alpha\big)\, d\theta.$$

Define $\mathcal{S}_0 \subset \mathcal{S}$ to be the set of simple closed curves α for which $\ell_{q_n}(\alpha) \to 0$. Equivalently, $\alpha \in \mathcal{S}_0$ if and only if $\mathrm{i}(L_\infty, \alpha) = 0$. Let Z_0 be the subsurface of S that is filled by $\mathcal{S}_0$—that is, Z_0 is the largest (via containment, up to isotopy) π_1–injective subsurface Z with the property that every closed curve in S which cuts Z has positive intersection number with some curve in $\mathcal{S}_0$. If $Z_0 = S$, then there is a finite set $\alpha_1, \ldots, \alpha_k$ of curves in $\mathcal{S}_0$ such that $\sum \alpha_i$ is a binding current, and as L_∞ lies in the span of $\mathcal{ML}(S) \subset \mathsf{C}(S)$, we have

$$\sum \mathrm{i}(L_\infty, \alpha_i) > 0,$$

which is a contradiction. Therefore, Z_0 is a proper subsurface of S.

We observe that, for each $\alpha_0 \in \mathcal{S}_0$,

$$\frac{1}{2}\int_0^\pi \mathrm{i}(\alpha_0, f_\infty(\theta))\, d\theta = \mathrm{i}(L_\infty, \alpha_0) = 0.$$

Since f_∞ is continuous, this implies that $\mathrm{i}(\alpha_0, f_\infty(\theta)) = 0$ for every θ. That is, for every $\theta \in \mathbb{RP}^1$, the support of $f_\infty(\theta)$ can be homotoped to be disjoint from Z_0. Hence, $\mathrm{i}(\alpha, L_\infty) = 0$ for every essential curve in Z_0. However, the restriction of L_∞ to Z_0 may not be zero; for an annular component A of Z_0, the restriction of $f_\infty(\theta)$ to A may be a measured lamination that is supported on the core curve of A.

Now choose a component W of $S \setminus Z_0$. Define

$$D(W) = \left\{ \mathrm{i}\left(L_\infty, \frac{\alpha}{\ell_{q_0}(\alpha)}\right) \,\Big|\, \alpha \in \mathcal{S}(W) \right\}.$$

Observe that $D(W)$ is bounded, since $\{\frac{\alpha}{\ell_{q_0}(\alpha)}\}$ is precompact, being contained in the compact set

$$\{\lambda \in \mathcal{ML}(S) \,|\, \ell_{q_0}(\lambda) = 1\}.$$

We argue in two cases.

Case 1: $\inf(D(W)) > 0$.

In this case, we have a uniform lower bound for the q_n–length of any nonperipheral simple closed curve, and hence also any nonperipheral closed curve in W. Since W is a component of $S \setminus Z_0$, the q_n–lengths of the boundary curves of W go to zero. Therefore, after choosing a basepoint in W (away from the boundary) and passing to a subsequence, we can assume that $q_n|_W$ converges to a flat structure on W *geometrically*, that is, after remarking by a homeomorphism. (See Appendix A of [22] for a thorough discussion of the geometric topology on the space of quadratic differentials. In particular, McMullen establishes the existence of the relevant geometric limit in his Theorem A.3.1 for points in moduli space.) Since any given curve in W has a uniform upper bound to its q_n–length, we may assume that the remarking homeomorphisms are isotopic to the identity in W, and hence $q_n|_W$ converges to a flat structure on W (though not necessarily of unit area).

Case 2: $\inf(D(W)) = 0$.

In this case, we have a sequence of simple curves $\alpha_n \in \mathcal{C}(W)$ such that

$$\lim_{n\to\infty} \mathrm{i}\left(L_\infty, \frac{\alpha_n}{\ell_{q_0}(\alpha_n)}\right) = 0.$$

Since $\{\frac{\alpha_n}{\ell_{q_0}(\alpha_n)}\}$ is precompact, we may pass to a subsequence so that

$$\frac{\alpha_n}{\ell_{q_0}(\alpha_n)} \to \lambda,$$

for some lamination λ. The continuity of intersection number implies $\mathrm{i}(L_\infty, \lambda) = 0$.

We observe that λ has to fill W. To see this, let $W' \subset W$ be the subsurface filled by λ. Since $\mathrm{i}(L_\infty, \lambda) = 0$, it follows that $\mathrm{i}(f_\infty(\theta), \lambda) = 0$. Therefore, $\mathrm{i}(f_\infty(\theta), \partial W') = 0$ and hence $\mathrm{i}(L_\infty, \partial W') = 0$. Thus $\partial W' \in \mathcal{S}_0$ and $W = W'$. The support of L_∞ consists of geodesics having no transverse intersection with the support of λ. Therefore, the support of L_∞, restricted to W, equals the support of λ. That is, $L_\infty|_W$ is a (filling) measured lamination in W.

We have shown that L_∞ is a mixed structure (X, q, λ) where X is the union of all W as in Case 1, q is the limiting flat structure in X and λ is the union of limiting laminations in Case 2 and weighted curves from all the annular components A where the restriction of some $f_\infty(\theta)$ to A is nontrivial. Since

$$\mathrm{i}(L_\infty, L_\infty) = \lim_{n\to\infty} \mathrm{i}(L_{q_n}, L_{q_n}) = \pi/2,$$

the sum of the areas of the flat structures is 1.

To finish the proof, we show that any mixed structure $\eta = (X, q, \lambda)$ appears as the limit of a sequence of flat structures. The idea is to build the metric from q on X, by making small slits at the punctures and gluing in a sequence of metrics on the complement, limiting to λ and with area tending to zero.

First write the lamination λ as $\lambda = \lambda_0 + \lambda_1$, where λ_0 is supported on a disjoint union of simple closed curves, and λ_1 has support a lamination with no closed leaves. We can further decompose $\lambda_0 = \sum_i s_i \alpha_i$ for some $\alpha_i \in \mathcal{S}$ and $s_i > 0$. For each i and all $n \geq 0$, let $C_{i,n}$ be a Euclidean cylinder with height s_i and circumference $2/n^2$. Let Y be the subsurface filled by λ_1 (with boundary replaced by punctures) and let $q' \in \mathcal{Q}^1(Y)$ be any quadratic differential for which $\nu^0_{q'} = \lambda_1$. Consider the Teichmüller deformation $A_n q'$, where

$$A_n = \begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix}.$$

This tends to the vertical foliation of q' (which was chosen to be λ_1) by an argument which appears in Proposition 29.

Let Z be the union of the nonannular, non-pants components of $S \setminus (X \cup Y)$. Choose any quadratic differential $q'' \in \mathcal{Q}^1(Z)$ for which the vertical foliation is minimal (for simplicity).

Now we construct a flat structure q_n as follows. At each puncture in q that corresponds to an essential curve in S (that is, a boundary component of X in S) we cut open a slit of size $1/n^2$ emanating from the given puncture, in any direction. Similarly, letting

$$q'(n) = \frac{1}{n} A_n q' \qquad \text{and} \qquad q''(n) = \frac{1}{n} q'',$$

cut open slits of length $1/n^2$ along the vertical foliations of each, one starting at each of the punctures of Y and Z that correspond to essential curves in S. Note that since the vertical foliations of $q'(n)$ and $q''(n)$ are minimal, these constructions are possible. We glue these and the cylinders $\{C_{i,n}\}$ along their boundaries to recover the surface S with a quadratic differential q_n, which we scale to have unit norm (as n tends to infinity, the areas of $q'(n)$ and $q''(n)$ go to zero and the scaling factor tends to 1). We glue along the boundaries by a local isometry, and if we further require the relative twisting of q_0 and q_n along every gluing curve to be uniformly bounded, we obtain a sequence, as desired, limiting to η in $\mathcal{P}\mathsf{C}(S)$. □

A dimension count. The codimension of the Thurston boundary in $\overline{\mathcal{T}(S)}$ is one. Here, we can show that the codimension of $\partial\mathsf{Flat}(S)$ is three. To see this, first recall that for a connected surface S of genus g with n punctures, $\mathcal{T}(S)$ is $(6g+2n-6)$–dimensional. The space $\mathcal{Q}(S)$ of quadratic differentials on S has twice the dimension and $\mathsf{Flat}(S)$ is a quotient of $\mathcal{Q}(S)$ by an action of $\mathbb{C}$. Hence

$$\dim(\mathsf{Flat}(S)) = 12g + 4n - 14.$$

For any π_1–injective subsurface $Y \subset S$, we consider the subset $\partial_Y \subset \partial\mathsf{Flat}(S)$ consisting of those $\eta = (X, q, \lambda)$ for which $X = S \setminus Y$. Observe that $\partial\mathsf{Flat}(S)$ is a disjoint union of subsets of the form ∂_Y, as Y varies over subsurfaces of S. In the case that Y is an annulus with core curve α, we simply write $\partial_Y = \partial_\alpha$. Points in ∂_α are projective mixed structures of the form $w\alpha + L_q$, where $q \in \mathsf{Flat}(X)$ and the weights w on α are nonnegative numbers. We first compute the dimension of the sets ∂_α.

If α is a non-separating curve, then X is connected, has genus one less than S and has 2 extra punctures. That is,

$$\dim(\mathsf{Flat}(X)) = 12(g-1) + 4(n+2) - 14 = 12g + 4n - 18.$$

To recover the space ∂_α, we restore one extra dimension from the weight on α, so that $\dim(\partial_\alpha) = 12g+4n-17$, which has codimension three with respect to $\mathsf{Flat}(S)$.

Now let α be a separating curve. Then $X = X_1 \cup X_2$, where X_i is a surface of genus g_i with n_i punctures ($i = 1, 2$) so that $g = g_1 + g_2$ and $n = n_1 + n_2 + 2$. Therefore, $\mathcal{Q}(X)$ has dimension

$$(12g_1 + 4n_1 - 12) + (12g_2 + 4n_2 - 12) = 12g + 4(n+2) - 24 = 12g + 4n - 16.$$

The space $\mathsf{Flat}(X)$ is the quotient of $\mathcal{Q}(X)$ by scaling and rotation in each component, but the total area must be one in the end, giving

$$\dim(\mathsf{Flat}(X)) = \dim(\mathcal{Q}(X)) - 3 = 12g + 4n - 19.$$

The space ∂_α has one extra dimension from the weight on α and is $(12g+4n-18)$–dimensional. In the separating case, then, the codimension is four with respect to $\mathsf{Flat}(S)$.

It is not difficult to see that for larger-complexity subsurfaces $Y \subset S$, the subsets ∂_Y have higher codimension in $\mathsf{Flat}(S)$, since for any subsurface W,

$$\dim \mathcal{ML}(W) < \dim \mathsf{Flat}(W).$$

Since $\partial\mathsf{Flat}(S)$ is a countable union of sets of the form ∂_Y, each of which can be exhausted by compact (hence closed) sets, the dimension of $\partial\mathsf{Flat}(S)$ is the maximum dimension of any subset ∂_Y (by the Sum Theorem in [23]), which is therefore $12g + 4n - 16$. So we have seen that $\partial\mathsf{Flat}(S)$ has codimension three in $\overline{\mathsf{Flat}(S)}$.

7. Remarks and questions

7.1. **Rigidity for closed curves.** Though we have a complete description of rigidity for $\Sigma \subset \mathcal{S}$, the more general case of $\Sigma \subset \mathcal{C}$ is still open.

We have already seen a sufficient condition for $\Sigma \subset \mathcal{C}$ to be spectrally rigid over flat metrics: clearly if $\mathcal{PMF} \subset \mathcal{P}(\overline{\Sigma})$, then Σ is spectrally rigid because its lengths determine all those from $\mathcal{S}$. Here is a further observation.

Proposition 27. *If $\overline{\Sigma}$ has interior as a subset of $\mathsf{C}(S)$, then Σ is spectrally rigid over any class of metrics that embeds naturally into $\mathsf{C}(S)$.*

Proof. Fix a pair of currents ν_1,ν_2 and set

$$f(\mu) := \mathrm{i}(\nu_1, \mu) - \mathrm{i}(\nu_2, \mu).$$

Suppose there is an open set in $f^{-1}(0)$ containing a current μ_0. Let $\{\delta\}$ be the set of currents close to the identity in the metric on $\mathsf{C}(S)$ (defined in Theorems 11,15). Then for δ sufficiently close to the zero current, $f(\mu + \delta) = 0$, so $f(\delta) = 0$ by linearity of i. But every current is a multiple of a small current and f is linear, so this shows that ν_1 and ν_2 have the same intersection number with all of the elements of $\mathsf{C}(S)$. We can conclude that $\nu_1 = \nu_2$ by Otal's theorem. In fact, we have shown that intersections with any open set of currents suffice to separate points in $\mathsf{C}(S)$.

To apply this to a class of metrics such that $\mathcal{G}(S) \hookrightarrow \mathsf{C}(S)$ and $\mathrm{i}(L_\rho, \alpha) = \ell_\rho(\alpha)$, suppose that $\lambda_\Sigma(\rho) = \lambda_\Sigma(\rho')$. Letting

$$\nu_1 = L_\rho \qquad \text{and} \qquad \nu_2 = L_{\rho'},$$

we have $f(\mu) = 0$ for all $\mu \in \overline{\Sigma}$, which contains an open set by assumption. This then implies that $\rho = \rho'$. □

7.2. Remarks on the boundary of $\mathsf{Flat}(S)$.

Remark 28. We observe that Teichmüller geodesics behave well with respect to the compactification of $\mathsf{Flat}(S)$. For all the points along a Teichmüller geodesic, the vertical and horizontal foliations are constant, up to scaling. In this compactification, every geodesic limits to its vertical foliation.

Proposition 29. *Let $G\colon \mathbb{R} \to \mathcal{T}(S)$ be a Teichmüller geodesic, let q_t be the corresponding quadratic differential at time t and ν_0 be the initial vertical foliation at q_0. Then, considering ν_0 as an element of $\mathsf{C}(S)$, we have*

$$\frac{L_{q_t}}{e^t} \to \nu_0.$$

Proof. The flat length of a curve is less than the sum of its horizontal length and its vertical length and is larger than the minimum of the its horizontal and vertical lengths. That is, if μ_t and ν_t are the horizontal and the vertical foliation at q_t then for every $\alpha \in \mathcal{C}'(S)$ we have

$$\min\big(\mathrm{i}(\alpha, \nu_t), \mathrm{i}(\alpha, \mu_t)\big) \le \ell_{q_t}(\alpha) \le \mathrm{i}(\alpha, \nu_t) + \mathrm{i}(\alpha, \mu_t).$$

But $\mathrm{i}(\alpha, \nu_t) = e^t\, \mathrm{i}(\alpha, \nu_0)$ and $\mathrm{i}(\alpha, \mu_t) = e^{-t}\, \mathrm{i}(\alpha, \mu_0)$. Therefore,

$$\frac{\mathrm{i}(L_{q_t}, \alpha)}{e^t} = \frac{\ell_{q_t}(\alpha)}{e^t} \to \mathrm{i}(\alpha, \nu_0).$$

Theorems 10 and 14 assure us that a current is completely determined by these intersections. □

This proposition shows not only that points along a Teichmüller geodesic converge to a unique limit in $\partial\mathsf{Flat}(S) = \mathcal{P}\mathsf{Mix}(S)$, but also that different geodesic rays with a common basepoint have different limit points in the boundary (because they have different vertical foliations). This is in contrast with the situation for the Thurston boundary where both of the above statements are false (see [18] and [20]).

Remark 30. The boundary of $\mathsf{Flat}(S)$ described here and the Thurston boundary of Teichmüller space are compatible in a certain sense. Consider the projection

$$\sigma : \mathsf{Flat}(S) \to \mathcal{T}(S)$$

which sends a flat metric q to the hyperbolic metric in its conformal class. As flat structures degenerate to the boundary, the corresponding hyperbolic metrics accumulate in $\mathcal{PML}$. The following proposition describes the relationship between the limiting structures: they have zero intersection number. The results on Teichmüller geodesics in the previous remark illustrate a special case of this.

Proposition 31. *Let q_n be a sequence of flat structures on S and $\sigma_n = \sigma(q_n)$. Assume that $\sigma_n \to \mu$ in the Thurston compactification and $q_n \to \eta$ in $\mathcal{P}\mathsf{C}(S)$, where μ is a geodesic lamination and η is a mixed structure in $\partial\mathsf{Flat}(S)$. Then*

$$\mathrm{i}(\mu, \eta) = 0.$$

Proof. We suppose that $s_n q_n \to \eta$ as currents and $t_n \ell_{\sigma_n}(\nu) \to \mathrm{i}(\mu, \nu)$ for all $\nu \in \mathcal{ML}(S)$. Since the σ_n and q_n escape from $\mathcal{T}$ and $\mathsf{Flat}(S)$, respectively, we know that the t_n tend to zero and the s_n are bounded. There is a sequence of approximating laminations μ_n to σ_n such that $t_n\mu_n \to \mu$ in $\mathcal{ML}(S)$ and $\mathrm{i}(\mu_n, \nu) \le \ell_{\sigma_n}(\nu)$ for all $\nu \in \mathcal{ML}(S)$; see [1, Exposé 8]. Then we have

$$\begin{aligned}
\mathrm{i}(\mu, \eta) &= \lim_{n\to\infty} \mathrm{i}(t_n\mu_n, s_n L_{q_n}) \\
&= \lim_{n\to\infty} \frac{1}{2}\int_0^\pi \mathrm{i}\left(t_n\mu_n, s_n\nu^\theta_{q_n}\right)\, d\theta \\
&\le \lim_{n\to\infty} \frac{1}{2}\int_0^\pi t_n\ell_{\sigma_n}\left(s_n\nu^\theta_{q_n}\right)\, d\theta.
\end{aligned}$$

We also have that $\ell_{\sigma_n}(\nu^\theta_{q_n})$ is bounded above by $\sqrt{A\cdot \mathrm{Ext}_{[\sigma_n]}(\nu^\theta_{q_n})}$, where A is the σ_n-area of S, which is a constant. This is true for simple closed curves by definition of extremal length, and holds for laminations because both hyperbolic length and extremal length extend continuously to $\mathcal{ML}(S) = \mathcal{MF}(S)$; see [17]. Furthermore, extremal length of $\nu^\theta_{q_n}$ is realized in the quadratic differential metric for which the foliation is straight, namely q_n. Finally, since $\ell_{q_n}(\nu^\theta_{q_n}) = 1$ and since the product $t_n s_n$ tends to zero, we conclude that $\mathrm{i}(\mu, \eta) = 0$, as desired. □

Note that if ρ is any metric in the conformal class of q to which a current L_ρ can be naturally associated, the extremal length argument gives us that $\mathrm{i}(L_\rho, L_q) \le \frac{\pi}{2}\sqrt{A}$, for A the ρ-area of the surface. This gives an even simpler proof of the previous theorem for the case of closed surfaces S by taking ρ to be the hyperbolic metric in the conformal class of q. Furthermore, we also have the following interesting inequality:

$$\mathrm{i}(L_q, L_{q'}) \le 1$$

where q and q' are any two flat metrics in the same fiber over $\mathcal{T}(S)$.

Remark 32. The boundary for $\mathsf{Flat}(S)$ can be used to construct a boundary for $\mathcal{Q}^1(S)$. We have shown that, for a sequence q_n of quadratic differentials, after taking a subsequence, not only L_{q_n} converge in $\mathsf{C}(S)$, but Proposition 26 shows that the maps $f_n(\theta)$ converge uniformly to a map f_∞, after appropriately scaling. One can equip the space

$$\left\{ (\mu, f) \,\middle|\, \mu \in \mathsf{C}(S), \quad f\colon\, \mathbb{RP}^1 \to \mathcal{ML}(S) \quad \text{continuous} \right\}$$

with the product topology from $\mathsf{C}(S)$ in one factor and uniform convergence in the other. Then the map $q_n \mapsto (L_{q_n}, f_n)$ is an embedding and has compact closure in the projectivization. However, it seems difficult to describe which pairs (μ, f) appear in the boundary of $\mathcal{Q}^1(S)$.

7.3. **Unmarked length spectrum does not suffice.** The Sunada construction of distinct isospectral hyperbolic surfaces, originally put forward in [31], is easily applied to metrics in $\mathsf{Flat}(S)$. We briefly sketch the idea.

Sunada constructs non-isometric hyperbolic surfaces S_1,S_2 covering a common S by choosing "almost-conjugate" subgroups Γ_1, Γ_2 of $\pi_1(S)$ and lifting to corresponding covers. (Almost-conjugacy means that each conjugacy class of $\pi_1(S)$ intersects the two subgroups in the same number of elements.) If a flat metric q is placed on S, then the argument that its lifts q_1, q_2 are iso-length-spectral runs exactly as for the hyperbolic metrics: $\Lambda_{\mathcal{C}}(q_1) = \Lambda_{\mathcal{C}}(q_2)$, because an element of $\pi_1(S)$ conjugating $\gamma_1 \in \Gamma_1$ to $\gamma_2 \in \Gamma_2$ associates a geodesic of q_1 for which the associated deck transformation is γ_1 to an equal-length q_2–geodesic by acting on the lift to $\tilde{S}$. (See [6] for a careful discussion.)

The key in using the Sunada construction is therefore to find examples for which the metrics on S_1 and S_2 are not isometric, but such choices of hyperbolic metrics on S are in fact generic. Now put a flat metric q on S in the conformal class of such a hyperbolic metric, and lift it to flat metrics q_i on S_i. If q_1 is isometric to q_2, then they are conformally equivalent, so the corresponding hyperbolic metrics are equal, a contradiction. Thus there is a ready supply of examples of distinct flat metrics for which $\Lambda_{\mathcal{C}}(q_1) = \Lambda_{\mathcal{C}}(q_2)$.

Note that this argument is for the unmarked length spectrum $\Lambda_{\mathcal{C}}$ of all closed curves; the counts of lifts in the Sunada construction are not sensitive to whether curves are simple. The question of whether there are distinct flat surfaces with equal unmarked length spectrum for the simple closed curves $\mathcal{S}$ remains open.

8. Appendix: More building blocks.

Here we sketch the construction of the remaining basic building blocks needed to carry out the proof of Proposition 20 for a general surface S with $\xi(S) \geq 2$. The building blocks are surfaces $\Sigma_{g,n,b}$ where g is the genus, n, is the number of punctures/marked points, and b is the number of boundary components. If we let d denote the dimension of the space of metrics we construct on $\Sigma_{g,n,b}$, then the resulting pairs $(\Sigma_{g,n,b}, d)$ are:

$$(\Sigma_{1,0,2}, 2)\,,\ (\Sigma_{1,0,1}, 2)\,,\ (\Sigma_{1,1,1}, 2)\,,\ (\Sigma_{0,2,1}, 0)\,,\ (\Sigma_{0,3,1}, 2)\,,\ (\Sigma_{0,4,1}, 2)\,,\ (\Sigma_{0,4,2}, 3).$$

The case of $\Sigma_{1,0,2}$ was discussed in Section 4. Each building block will come equipped with a train track that carries the boundary, and when the building blocks are assembled to construct the surface S, the train tracks assemble to a complete recurrent train track. The family of metrics for each will keep the boundary length fixed, so that the deformations can be carried out independently on each piece. Gluing together the deformations is carried out in a fashion similar to that used for the closed case in Section 4.

By gluing the pieces above, one can construct flat structures and magnetic train-tracks on any surface with $\xi \geq 2$. The details (which pieces are needed for which surface) are left to the reader. Essentially, one can attach copies of $\Sigma_{1,0,2}$ (to add one genus) and $\Sigma_{0,4,2}$ (to add four punctures) along their boundaries to obtain a surface with two boundary components which has almost all the required genus and number of punctures. One then caps off the two boundaries with the appropriate pieces to obtain the desired surface. The resulting flat structure, q, and magnetic train track, τ, has a deformation where the length of every curve in τ remains constant. The dimension of the this deformation space is at least equal to the sum of the deformations allowable deformations for the pieces involved. We should then subtract one to restrict the deformation to quadratic differentials of area 1.

8.1. $(\Sigma_{1,0,1}, 2)$**.** The topological picture of $\Sigma_{1,0,1}$ together with its train track are shown on the left in Figure 10.

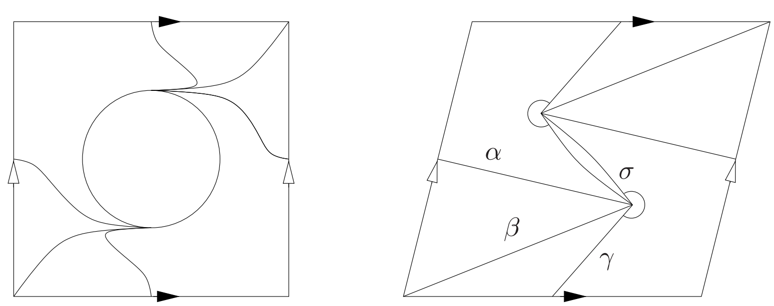

Figure 10. The topological picture of $\Sigma_{1,0,1}$ and its train track on the left and the metric picture on the right. The angles formed by the segments which are at least π are indicated in the picture.

The arcs in the boundary of the square are identified in pairs as indicated by the arrows. The generic metric in the deformation family is shown on the right in Figure 10 and is described as follows. Starting with a parallelogram having one horizontal side and one skew side with positive slope, we identify the opposite sides by a translation as indicated by the arrows. Next we cut a slit in a geodesic arc σ in the parallelogram, and we assume that σ has negative slope. This produces a metric version of the topological surface, and the geodesic version of the train track is obtained by adding the arcs α, β, γ as indicated.

If we require the boundary length to be fixed, so the length of σ is fixed, then the dimension of the space of all such metrics is 4: there are 3 dimensions for the parallelogram and one for the angle σ makes with the horizontal side. We now wish to impose constraints which guarantee that the change in lengths of the branches can be distributed to the switches in such a way that at each switch the increase in the lengths of the incoming branches is equal to the decrease in lengths of the outgoing branches. In this case, one checks that this can only be accomplished if each of the lengths of α, β, γ change by the same amount. This imposes two conditions: the difference in lengths of α and β is constant, and the difference in lengths of β and γ is constant. This cuts the dimension of the deformation space down by two, resulting in the 2–dimensional space of deformations that was claimed. It is interesting to note that in this case, there are nontrivial deformations for which the length vector on the train track itself remains constant.

8.2. $(\Sigma_{1,1,1}, 2)$**.** This building block is obtained by a minor modification of the previous one; see Figure 11. We leave the details to the reader, but point out one new feature in this example not present in the previous two pieces. Namely, the map $f : (\hat{S}, P) \to (\hat{S}, P)$ in the definition of a magnetic train track cannot be taken to be a homeomorphism. This is because the small branch that partially surrounds the puncture is collapsed to a point—the length vector assigns this branch zero length.

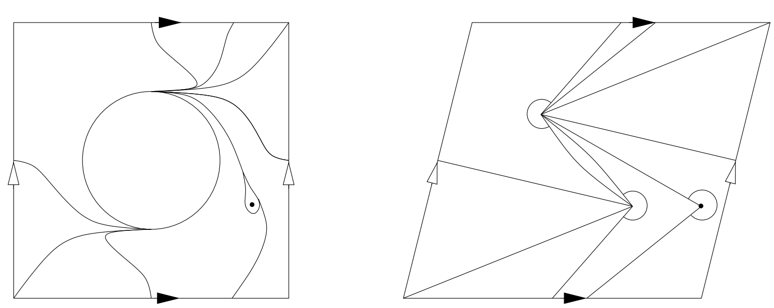

FIGURE 11. The case $\Sigma_{1,1,1}$ is a minor variation of $\Sigma_{1,0,1}$ shown in Figure 10.

8.3. $(\Sigma_{0,2,1}, 0)$**.** For this building block, the metric picture degenerates completely to an arc and there is "no room" to construct any deformations; see Figure 12. This piece is used to cap off boundary components. Alternatively, in capping off a boundary component, the metric effect is simply to glue the boundary component to itself.

8.4. $(\Sigma_{0,3,1}, 1)$**.** The generic metric is obtained from a parallelogram by identifying the arcs in the sides as indicated by the arrows in Figure 13 via an appropriate semi-translation, then cutting open a slit in the interior emanating from one of the marked points. The small loop branches of the train track are assigned zero

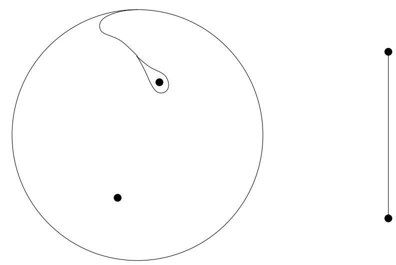

FIGURE 12. The metric version for $\Sigma_{0,2,1}$ degenerates.

length, and the three main branches (not in the boundary) are represented by the darkened arcs in the metric picture. A dimension count as above reveals that the deformation space has dimension 2.

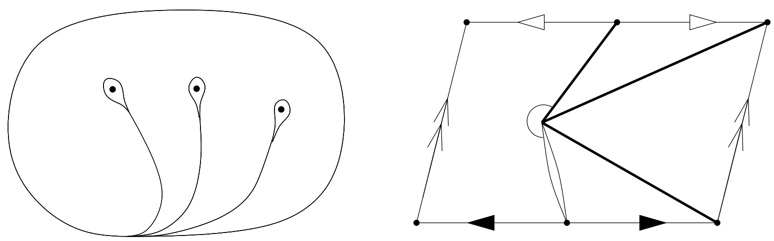

FIGURE 13. The dark lines in the metric picture for $\Sigma_{0,3,1}$ represent the image of the three main branches of the train track.

8.5. $(\Sigma_{0,4,1}, 1)$. This building block is obtained from the previous one in a similar fashion to the way $\Sigma_{1,1,1}$ is obtained from $\Sigma_{1,0,1}$; see Figure 14. We leave the details to the reader.

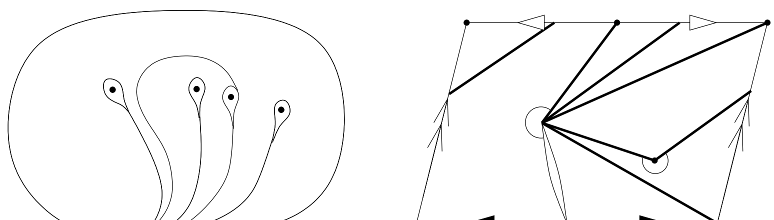

FIGURE 14. Adding another puncture to $\Sigma_{0,3,1}$ to produce $\Sigma_{0,4,1}$.

8.6. $(\Sigma_{0,4,2}, 3)$. The metric picture is formed from a parallelogram with sides identified as illustrated, then slit open along two equal-length arcs as shown. We have labeled some of the branches of the train track in Figures 15 and 16.

The space of allowable deformations has dimension 3. To see this, first note that to properly distribute length changes at the switches, the lengths of the arcs are

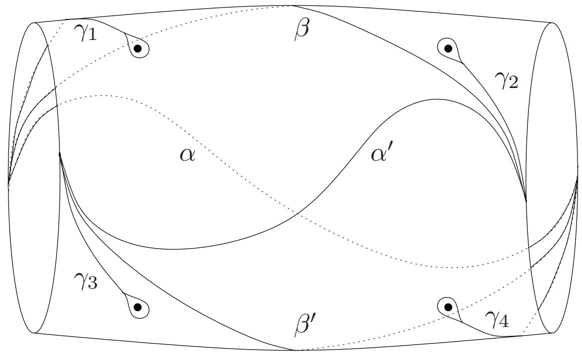

FIGURE 15. The topological picture of $\Sigma_{0,4,2}$ together with its train track.

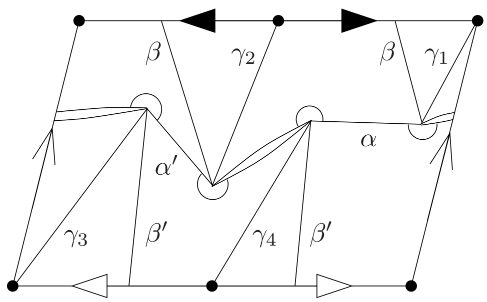

FIGURE 16. The metric picture of $\Sigma_{0,4,2}$.

allowed to vary according to the following:

α	α'	β	β'	γ_1	γ_2	γ_3	γ_4
$+\epsilon+\delta$	$+\epsilon+\delta$	$+\epsilon+\delta$	$+\epsilon+\delta$	$+\epsilon$	$+\delta$	$+\epsilon$	$+\delta$

To see that these variations are indeed possible (for small ϵ and δ), we again appeal to a dimension count. The space of parallelograms with a pair of slits of fixed, equal length is $3+3+3=9$ dimensions. The 9-dimensional parameter space is subject to 6 equations derived from the geometry, leaving 3 degrees of freedom.

REFERENCES

[1] *Travaux de Thurston sur les surfaces*, volume 66 of *Astérisque*. Société Mathématique de France, Paris, 1979. Séminaire Orsay, With an English summary.

[2] Roger Alperin and Hyman Bass. Length functions of group actions on Λ-trees. In *Combinatorial group theory and topology (Alta, Utah, 1984)*, volume 111 of *Ann. of Math. Stud.*, pages 265–378. Princeton Univ. Press, Princeton, NJ, 1987.

[3] Joan S. Birman and Caroline Series. Geodesics with bounded intersection number on surfaces are sparsely distributed. *Topology*, 24(2):217–225, 1985.

[4] Francis Bonahon. Bouts des variétés hyperboliques de dimension 3. *Ann. of Math. (2)*, 124(1):71–158, 1986.

[5] Francis Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92(1):139–162, 1988.

[6] Peter Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.

[7] M. M. Cohen, M. Lustig, and M. Steiner. **R**-tree actions are not determined by the translation lengths of finitely many elements. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 183–187. Springer, New York, 1991.
[8] C. Croke, A. Fathi, and J. Feldman. The marked length-spectrum of a surface of nonpositive curvature. *Topology*, 31(4):847–855, 1992.
[9] Christopher B. Croke. Rigidity for surfaces of nonpositive curvature. *Comment. Math. Helv.*, 65(1):150–169, 1990.
[10] Marc Culler and John W. Morgan. Group actions on **R**-trees. *Proc. London Math. Soc. (3)*, 55(3):571–604, 1987.
[11] Albert Fathi. Le spectre marqué des longueurs des surfaces sans points conjugués. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(9):621–624, 1989.
[12] Frederick P. Gardiner and Nikola Lakic. *Quasiconformal Teichmüller theory*, volume 76 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.
[13] Frederick P. Gardiner and Howard Masur. Extremal length geometry of Teichmüller space. *Complex Variables Theory Appl.*, 16(2-3):209–237, 1991.
[14] Ursula Hamenstädt. Length functions and parameterizations of Teichmüller space for surfaces with cusps. *Ann. Acad. Sci. Fenn. Math.*, 28(1):75–88, 2003.
[15] Ursula Hamenstädt. Parametrizations of Teichmüller space and its Thurston boundary. In *Geometric analysis and nonlinear partial differential equations*, pages 81–88. Springer, Berlin, 2003.
[16] Sa'ar Hersonsky and Frédéric Paulin. On the rigidity of discrete isometry groups of negatively curved spaces. *Comment. Math. Helv.*, 72(3):349–388, 1997.
[17] Steven P. Kerckhoff. The asymptotic geometry of Teichmüller space. *Topology*, 19(1):23–41, 1980.
[18] Anna Lenzhen. Teichmüller geodesics that do not have a limit in PMF. *Geom. Topol.*, 12(1):177–197, 2008.
[19] Gilbert Levitt. Foliations and laminations on hyperbolic surfaces. *Topology*, 22(2):119–135, 1983.
[20] Howard Masur. On a class of geodesics in Teichmüller space. *Ann. of Math. (2)*, 102(2):205–221, 1975.
[21] Howard Masur. Closed trajectories for quadratic differentials with an application to billiards. *Duke Math. J.*, 53(2):307–314, 1986.
[22] Curt McMullen. Amenability, Poincaré series and quasiconformal maps. *Invent. Math.*, 97(1):95–127, 1989.
[23] Jun-iti Nagata. *Modern dimension theory*, volume 2 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, revised edition, 1983.
[24] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Ann. of Math. (2)*, 131(1):151–162, 1990.
[25] Athanase Papadopoulos, editor. *Handbook of Teichmüller theory. Vol. I*, volume 11 of *IRMA Lectures in Mathematics and Theoretical Physics*. European Mathematical Society (EMS), Zürich, 2007.
[26] R. C. Penner and J. L. Harer. *Combinatorics of train tracks*, volume 125 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992.
[27] K. Rafi. A characterization of short curves of a Teichmüller geodesic. *Geometry and Topology*, 9:179–202, 2005.
[28] Paul Schmutz. Die Parametrisierung des Teichmüllerraumes durch geodätische Längenfunktionen. *Comment. Math. Helv.*, 68(2):278–288, 1993.
[29] John Smillie and Karen Vogtmann. Length functions and outer space. *Michigan Math. J.*, 39(3):485–493, 1992.
[30] K. Strebel. *Quadratic differentials*, volume 5 of *A series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1980.
[31] Toshikazu Sunada. Riemannian coverings and isospectral manifolds. *Ann. of Math. (2)*, 121(1):169–186, 1985.

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