Honors question 1: Complex numbers.

We defined the set of complex numbers by

$$\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\ ,\ i^2 = -1\}$$

We noted that we can add and multiply complex numbers by

$$(x + iy) + (u + iv) = (x + u) + i(y + v)$$

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu)$$

This is just formal multiplication applying $i^2 = -1$.

Note that the real numbers form a subset of the complex numbers where we write

$$x = x + i0 \text{ for any } x \in \mathbb{R}$$

As such, we note that $0, 1 \in \mathbb{C}$, and they play a special role:

(a.) *Verify that if z is any complex number, then*

$$z + 0 = z \textit{ and } z \cdot 1 = z$$

There is an operation on complex numbers called *conjugation*, denoted by a "bar on top" and defined by

$$\overline{x + iy} = x - iy$$

Every complex number z has an *additive inverse* denoted $-z$:

$$z + (-z) = 0$$

Here, if $z = x + iy$, then $-z = -x - iy$. Similarly, every complex number $z \neq 0$ has a *multiplicative inverse* denoted $\frac{1}{z}$:

$$z \cdot \frac{1}{z} = 1$$

If $x \in \mathbb{R}$, then $\frac{1}{x}$ (as a complex number) is just the usual reciprocal of x (e.g. $\frac{1}{2}$ is just "one half").

(b.) *Find an expression for $\frac{1}{z}$ in terms of x and y if $z = x + iy$. Also, find an expression for $\frac{1}{z}$ in terms of z and $\overline{z}$.*

Formally define:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

for any $\theta \in \mathbb{R}$.

(c.) *Verify that for any $\theta \in \mathbb{R}$ we have*

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Does this look familiar?

(d.) *Assuming* $e^x e^{iy} = e^{x+iy}$ *for every* $x, y \in \mathbb{R}$, *verify that*

$$e^z e^w = e^{z+w}$$

for any $z, w \in \mathbb{C}$.

(e.) *Find all complex numbers* $z = x + iy$ *for which* $z^6 = 1$, *and express each as* $x + iy$ *for real numbers* x *and* y, *with* x *and* y *expressed in terms of square roots. Plot these 6 numbers in the complex plane.*