Honors question 2: Approximating $n!$.

(a.) Suppose that f is a differentiable function on the interval $[0,1]$. Show that

$$\int_0^1 f(x)dx = \frac{f(0)+f(1)}{2} - \int_0^1 f'(x)\left(x-\frac{1}{2}\right)dx.$$

(b.) Suppose f is twice differentiable on $[0,1]$ (i.e. f' and f'' both exist). Show that

$$\int_0^1 f(x)dx = \frac{f(0)+f(1)}{2} + \int_0^1 f''(x)\left(\frac{x^2-x}{2}\right)dx$$

(c.) Find the absolute maximum and minimum of the function $\frac{x^2-x}{2}$ on the interval $[0,1]$.

(d.) Suppose f is twice differentiable on $[0,1]$ and there exists $M \geq 0$ so that $|f''(x)| \leq M$ for every x in $[0,1]$. Prove

$$\left|\int_0^1 f(x)dx - \frac{f(0)+f(1)}{2}\right| \leq \frac{M}{8}$$

Hint: Recall that if F is any integrable function on $[a,b]$ with $|F(x)| \leq M$, then

$$\left|\int_a^b F(x)dx\right| \leq M|b-a|$$

(e.) Let k be a positive integer and use the above to prove

$$\left|\int_0^1 \ln(x+k)dx - \frac{\ln(k)+\ln(k+1)}{2}\right| \leq \frac{1}{8k^2}$$

(f.) Prove

$$\int_0^1 \ln(x+k)dx = \int_k^{k+1} \ln(x)dx.$$

(g.) Prove

$$\left|\left(\int_1^{n+1} \ln(x)dx - \sum_{k=1}^{n} \ln(k)\right) - \frac{\ln(n+1)}{2}\right| \leq \sum_{k=1}^{n} \frac{1}{8k^2}$$

Hint:

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|.$$

(h.) Prove that there is a positive constant $c \geq 1$ so that for all $n \geq 1$

$$\frac{1}{c} \leq \frac{e^n n!}{(n+1)^{n+\frac{1}{2}}} \leq c.$$

Hint: Evaluate the integral in part (g.) and exponentiate.

(i.) (**Challenge**) Prove that the sequence

$$\left\{\frac{e^n n!}{(n+1)^{n+\frac{1}{2}}}\right\}_{n=1}^{\infty}$$

converges to a positive number.