

Honors question 3: Complex numbers (revisited).

Arithmetic of complex numbers

Recall that the complex numbers are formally defined as

$$\mathbb{C} = \{a + bi\}$$

where a and b can be any real numbers and i is treated as a variable (so we can identify the complex numbers with the set of linear polynomials with real coefficients). The numbers a and b are called the **real** and **imaginary parts** of $a + bi$, respectively. We view the real numbers as a subset of the complex numbers by identifying the real number a with $a + 0i$. The **imaginary numbers** are the numbers of the form $0 + bi$ for any nonzero real number b .

Addition in the complex numbers is the same as in the set of linear polynomials and is given by

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

Multiplication is defined by

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

and can be thought of as multiplication of polynomials, subject to the condition that $i^2 = -1$. Indeed, notice that if we consider the imaginary number $i = 0 + i$, then we have $i^2 = -1 = -1 + 0i$.

We sometimes write our complex numbers as $z = a + bi$. So, if we say that z is a complex number, then it has real and imaginary parts a and b , respectively. We define the **conjugate** of z by

$$\bar{z} = a - bi.$$

- (a.) Check that $z\bar{z}$ is a positive real number.
- (b.) Check that the real and imaginary parts of z are given by

$$\frac{z + \bar{z}}{2} \quad \text{and} \quad \frac{i(\bar{z} - z)}{2},$$

respectively.

According to part (a.), we can define the **modulus** of a complex number z to be the positive square root

$$|z| = \sqrt{z\bar{z}}.$$

We write the multiplicative inverse of a nonzero complex number z as $1/z$, just as in real case (so, $z \cdot 1/z = 1 = 1 + 0i$). Note that $1/z = (1/|z|^2) \cdot \bar{z}$. Division z/w is just multiplication by the multiplicative inverse $z/w = z \cdot 1/w$.

Geometry of complex numbers

We can visualize the complex numbers by identifying the complex number $a + bi$ with a point (a, b) in the plane \mathbb{R}^2 . Then observe that $|z|$ is simply the distance to the origin in the plane.

- (c.) Let z be a complex number. Verify that there is a real number θ so that

$$z = |z|(\cos(\theta) + i \sin(\theta)).$$

- (d.) Verify that if z is not zero, then the number θ from part (b) is unique up to integral multiples of 2π . That is, if we write

$$z = |z|(\cos(\theta) + i \sin(\theta)) = |z|(\cos(\theta') + i \sin(\theta'))$$

for some real numbers θ and θ' , then $\theta - \theta' = 2k\pi$ for some integer k .

(e.) If $z = |z|(\cos(\theta) + i \sin(\theta))$ and $w = |w|(\cos(\psi) + i \sin(\psi))$, then verify that

$$z \cdot w = |z||w|(\cos(\theta + \psi) + i \sin(\theta + \psi)).$$

(f.) If $z = |z|(\cos(\theta) + i \sin(\theta))$, verify that for any non-negative integer k , we have

$$z^k = |z|^k(\cos(k\theta) + i \sin(k\theta)).$$

(g.) Use part (e) to verify that for any positive integer n and real numbers a_0, \dots, a_n , if $z = |z|(\cos(\theta) + i \sin(\theta))$, then we have

$$\sum_{k=0}^n a_k z^k = \sum_{k=0}^n a_k |z|^k \cos(k\theta) + i \sum_{k=0}^n a_k |z|^k \sin(k\theta).$$

(h.) Check that if

$$\sum_{k=0}^{\infty} a_k x^k$$

is a power series with radius of convergence $0 < R \leq \infty$, then for any complex number $z = |z|(\cos(\theta) + i \sin(\theta))$ with $|z| < R$, then the two series

$$\sum_{k=0}^{\infty} a_k |z|^k \cos(k\theta) \text{ and } \sum_{k=0}^{\infty} a_k |z|^k \sin(k\theta)$$

converge absolutely.

According to parts (g) and (h), if we are given a power series $\sum_{k=0}^{\infty} a_k x^k$ with a radius of convergence $0 < R \leq \infty$, and if $|z| < R$, then we can define

$$\sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} a_k |z|^k \cos(k\theta) + i \sum_{k=0}^{\infty} a_k |z|^k \sin(k\theta).$$

In particular, since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for any real number x , we can define

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

for any complex number z .

(i.) Using the power series representations for $\sin(\theta)$ and $\cos(\theta)$, check that

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

In particular, this gives a very compact form for complex numbers

$$z = |z|e^{i\theta}$$

and their multiplication takes a simple form:

$$z \cdot w = |z|e^{i\theta} \cdot |w|e^{i\psi} = |z||w|e^{i(\theta+\psi)}.$$

Using part (b), you should compare (i) with $\cosh(\theta)$ and $\sinh(\theta)$.

(j.) Finally, check that for any two complex numbers z and w , we have

$$e^{z+w} = e^z e^w.$$