# Honors question 4: Continued fractions.

Recall that the rational numbers, denoted $\mathbf{Q}$, are precisely those numbers that can be expressed as ratios of integers, $\frac{p}{q}$. Every rational number $\frac{p}{q} \in \mathbf{Q}$ has a *continued fraction expansion*:

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \cdots \frac{1}{a_n}}}}}$$

where $a_1, a_2, ..., a_n \in \mathbf{Z}$ with $a_2, ..., a_n$ positive. For example, consider the rational number $\frac{19}{7}$. This has continued fraction expansion given by:

$$2 + \cfrac{1}{1 + \frac{1}{2 + \frac{1}{2}}}$$

since

$$2 + \cfrac{1}{1 + \frac{1}{2+\frac{1}{2}}} = 2 + \cfrac{1}{1 + \frac{1}{\frac{5}{2}}} = 2 + \cfrac{1}{1 + \frac{2}{5}} = 2 + \cfrac{1}{\frac{7}{5}} = 2 + \frac{5}{7} = \frac{19}{7}$$

**(a.)** Find the continued fraction expansion for the rational numbers $\frac{81}{35}$, $\frac{237}{29}$, $\frac{1376}{231}$. Perhaps the following equations would be helpful to get you started on the first one:

$$81 = 2 \cdot (35) + 11$$

$$35 = 3 \cdot (11) + 2$$

$$11 = 5 \cdot (2) + 1$$

$$2 = 2 \cdot (1)$$

This notion of a continued fraction expansion for a rational number also makes sense for any real number, but now the sequence is no longer finite. More precisely, given any real number $x \in \mathbf{R}$, there is an infinite sequence $\{a_n\}_{n=1}^{\infty}$, where each $a_n \in \mathbf{Z}$ with $a_n > 0$ for $n \geq 2$, so that $x$ is the limit of the sequence of partial quotients

$$x = \lim_{n\to\infty} q_n$$

which are defined by

$$q_1 = a_1$$

$$q_2 = a_1 + \frac{1}{a_2}$$

$$q_3 = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3}}$$

$$q_4 = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}}$$

$$q_5 = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5}}}}$$

and so on, with the $n^{th}$ term given by

$$q_n = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \cdots \frac{1}{a_n}}}}}$$

It is not obvious that such a sequence would converge, but indeed, for *any* sequence $\{a_n\}_{n=1}^{\infty}$ with $a_n \in \mathbf{Z}$ for every $n$ and $a_n > 0$ for $n \geq 2$, this does indeed converge. For those that are interested in trying to see why this converges, let me give you a hint: Prove that the sequence is a *Cauchy sequence*—see the appendix in your book for this definition.

As we know, the rational numbers sit inside the real numbers $\mathbf{Q} \subset \mathbf{R}$. There is another set of numbers $\overline{\mathbf{Q}}$ which lies strictly between the rational and real numbers called the *algebraic numbers*, $\mathbf{Q} \subset \overline{\mathbf{Q}} \subset \mathbf{R}$ defined to be the set of numbers $x$ which satisfy a polynomial with integer coefficients:

$$x \in \overline{\mathbf{Q}} \quad \text{if and only if} \quad b_0 + b_1 x + b_2 x^2 + ... + b_n x^n = 0 \quad \text{for some } b_0, ..., b_n \in \mathbf{Z}$$

**(b.)** Verify that $\mathbf{Q} \subset \overline{\mathbf{Q}}$. That is, show that every rational number satisfies a polynomial equation with integer coefficients. Also check that $\sqrt{n} \in \overline{\mathbf{Q}}$ for every $n \in \mathbf{Z}$.

The algebraic numbers can be characterized in terms of continued fraction expansions. In particular if the continued fraction expansion of $x$ is repeating, then $x$ is an algebraic number. To say that the sequence is repeating means that it repeats a given finite string of integers over and over. E.g. $1, 3, 5, 2, 1, 3, 5, 2, 1, 3, 5, 2, .....$

**(c.)** Verify this fact for sequences that repeat strings of length 2, 3, and 4. That is, for sequences of the form

$$a, b, a, b, a, b, a, b, a, b, ...$$

and

$$a, b, c, a, b, c, a, b, c, a, b, c, ...$$

and

$$a, b, c, d, a, b, c, d, a, b, c, d, a, b, c, d, ...$$