

DG II: Problem Set 1

1.
 - a. Suppose that G is a Lie group with Lie algebra \mathfrak{g} . Prove that the restriction of \exp to any abelian subalgebra $\mathfrak{h} < \mathfrak{g}$, viewed as an additive group, is a homomorphism.
 - b. Prove that a connected abelian Lie group is isomorphic to $\mathbb{R}^k \times \mathbb{T}^n$, where \mathbb{T}^n is the n -torus.
2.
 - a. Suppose that $\alpha(t)$ and $\beta(t)$ are smooth curves in a Lie group G with $\alpha(0) = \beta(0) = e$. Set $\sigma(t) = \alpha(t)\beta(t)$. Prove that $\sigma'(0) = \alpha'(0) + \beta'(0)$.
 - b. Suppose now that $\alpha(t)$ and $\beta(t)$ are any smooth curves in a Lie group G defined on a neighborhood of 0. Set $\sigma(t) = \alpha(t)\beta(t)$. Prove

$$\sigma'(0) = d(r_{\beta(0)})_{\alpha(0)}(\alpha'(0)) + d(\ell_{\alpha(0)})_{\beta(0)}(\beta'(0))$$

3. Let $A \in \mathfrak{gl}_n(\mathbb{C})$.

- a. Prove that the formal series

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \dots$$

converges.

- b. Prove that for any $B \in \text{GL}_n(\mathbb{C})$ we have

$$Be^AB^{-1} = e^{BAB^{-1}}.$$

- c. Prove that $\det(e^A) = e^{\text{Trace}(A)}$.
- d. Prove that $e^{A+B} = e^A e^B$, for any $B \in \mathfrak{gl}_n(\mathbb{C})$ with $AB = BA$.
- e. Prove that e^{tA} is a smooth curve through I with derivative A at I .
- f. Conclude that $\exp(A) = e^A$.

4. Prove that the Maurer–Cartan form on $\text{GL}_n(\mathbb{R})$ is given by

$$g^{-1}dg.$$

(You'll have to explain what this means!)

5. Prove that Lie groups have no nontrivial small subgroups: If G is a Lie group, then there exists a neighborhood U of the identity e so that if $H \subset U$ is a subgroup of G , then $H = \{e\}$.
6. The *Killing form* on the Lie algebra \mathfrak{g} of a Lie group G is defined by

$$B(\xi, \eta) = \text{Trace}(\text{ad}_\xi \text{ad}_\eta).$$

- a. Prove that B is symmetric bilinear form on \mathfrak{g} .
- b. Choose a basis ξ_1, ξ_2, ξ_3 for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ and compute the matrix $B(\xi_i, \xi_j)$.

(a Lie algebra is called *semi-simple* if B is nondegenerate).