DG II: Problem Set 2

1. Let M be a smooth manifold and $\pi : T^*M \to M$ the cotangent bundle.

   a. Define a 1-form θ on T^*M by $\theta_\omega = \pi^*(\omega)$ for $\omega \in T^*M$ and check that it is smooth (hint: local coordinates $x^1, ..., x^n$ on M determine local coordinates $x^1, ..., x^n, y_1, ..., y_n$ on T^*M).

   b. Prove that $\omega = d\theta$ is a symplectic form on the **manifold** T^*M (a nondegenerate, closed 2-form).

2. Given a vector bundle $E \to M$ and a subbundle $E' \to M$, define a quotient bundle $E/E' \to M$, whose fibers $(E/E')_m = E_m/E'_m$.

3. If $E \to M$ is a rank k complex vector bundle, we can view it as a rank $2k$ real vector bundle and take its complexification $E_{\mathbb{C}} \to M$ (for this problem, it's easier to consider the complex structure on E as given by $J_0 \in \Gamma(End_{\mathbb{R}}(E))$ and let $J_1 = -J_0$ be the complex structure on $\overline{E}$).

   a. Prove that the inclusion $M \times \mathbb{R} \to M \times \mathbb{C}$ induces an injective bundle map $E \to E_{\mathbb{C}}$ of real vector bundles.

   b. Show that the complex structure $J_0 \in \Gamma(End_{\mathbb{R}}(E))$ determines a section $J \in \Gamma(End_{\mathbb{C}}(E_{\mathbb{C}}))$ with $J^2 = -1$.

   c. Prove that the pointwise eigenspaces of J induce on $E_{\mathbb{C}}$ the structure of a direct sum
   $$E_{\mathbb{C}} \cong E^h_{\mathbb{C}} \oplus E^{\overline{h}}_{\mathbb{C}}$$

   d. Find complex bundle isomorphisms $E \to E^h_{\mathbb{C}}$ and $\overline{E} \to E^{\overline{h}}_{\mathbb{C}}$.

4. Let S^n be the n–sphere. Prove that $TS^n \oplus (S^n \times \mathbb{R})$ is a trivial bundle.

5. Construct a principal S^1–bundle over $\mathbb{CP}^n$ with total space S^{2n+1}.

6. Construct a principal $U_n(\mathbb{C})$–bundle over the sphere S^{2n+1} with total space $U_{n+1}(\mathbb{C})$.